J. Partial Diff. Eqs. **15**(2002), 28–38
©International Academic Publishers Vol.15, No.4

# A HARDY TYPE INEQUALITY AND INDEFINITE EIGENVALUE PROBLEMS ON THE HOMOGENEOUS GROUP*

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(Received Apr. 4, 2002; revised Aug. 27, 2002)

**Abstract** In this paper we present a Hardy type inequality and a Picone type identity for the real sub-Laplacian on the homogeneous group. The existence of the indefinite eigenvalue problem and the simplicity of the principal eigenvalue are proved.

**Key Words** Hardy type inequality; Picone type identity; indefinite eigenvalue problem; homogeneous group.

**2000 MR Subject Classification** 35H20, 43A80.

**Chinese Library Classification** O175.9.

## 1. Introduction

Let $G = (R^N, \cdot)$ be a homogeneous group , $L$ be a real sub-Laplacian on $G$. The real sub-Laplacian on $G$ is

$$L = \sum_{j=1}^{p} X_j^2 . \tag{1}$$

The operator $L$ contains the real Kohn-Laplacian on the Heisenberg group as a particular case . A subelliptic regularity theory for $L$ was constructed by Folland in [?] and a fundamental solution of $L$ was found by Gallardo in [?] . The Liouville type theorems and Harnack type inequalities for $L$ were given by Bonfiglioli and Lanconelli in [?] . See also [?] for a general result .

In this paper we will establish a Hardy type inequality and a Picone type identity for $L$. As the applications , we prove the existence for the eigenvalue problem of $L$ with indefinite weights and the simplicity of the principal eigenvalue .

The plan of this paper is as follows . In Section 2 , we collect the basic and necessary material concerning the homogeneous group . We then present in Section 3 the Hardy type inequality for $L$. The proof uses a well-known representation formula for any

*This paper supported by National Natural Science Foundation of China and Natural Science Foundation of Shaanxi Province.

smooth function . In Section 4 , we examine the problem associated with the operator $L$

$$-Lu = \lambda g u \text{ in } G\,,\ u \to 0 \text{ as } |x| \to \infty\,, \tag{2}$$

where $g$ is a smooth bounded function that changes the sign, i.e., $g$ is an indefinite weight function. A principal eigenvalue of (2) is a $\lambda$ for which (2) has a positive solution. The indefinite eigenvalue problem in the Euclidean space has been treated extensively, see [?], [?], etc.. In [?], a Picone type identity for the Heisenberg Laplacian was proved. We present similar identity for $L$ and use it to check the simplicity of the principal eigenvalue in Section 5.

## 2. Notation and Recalls

For any $j = 1, \cdots, p$, let $X_j$ denote a first order differential operator with real smooth coefficients which is invariant with respect to left translations on a homogeneous group $G = (R^N, \cdot)$. The examples of homogeneous groups contain the Heisenberg group, the H-type group, etc., see [?], [?].

The Lie algebra $g$ generated by $\{X_1, \cdots, X_p\}$ is nilpotent , stratified and $N$-dimensional everywhere. If $\bigoplus_{j=1}^{\nu} g_j$ is the stratification of $g$, then $\{X_1, \cdots, X_p\}$ is a basis of $g_1$ .

A group of dilations on $G$ is denoted by $\delta_r$ , $r > 0$ , applying $R^N$ onto itself in the following way

$$\delta_r(x) = (r\,x^{(1)}, r^2 x^{(2)}, \cdots, r^{\nu} x^{(\nu)})\,, \tag{3}$$

where $x^{(j)}$ is the point of $R^{N_j}$ , $N_j's$ are positive integers such that $N_1 = p$ and $N_1 + \cdots + N_\nu = N$ .

The number $Q = \sum_{j=1}^{\nu} j\,N_j$ is the homogeneous dimension of $G$.

Lebesgue measure is invariant with respect to the left and right translations on $G$ . For any measurable set $E \subset R^N$ , it holds

$$meas(\delta_r(E)) = r^Q meas(E)$$

$X_j's$ are $\delta_r-$homogeneous of degree 1 and $X_j^\star = -X_j$ . Then $L = L^\star$ and $L$ is a divergence form operator .

A homogeneous norm $|\cdot|$ on $G$ is a function satisfying

1) $|\cdot| \in C^\infty(R^N \setminus \{0\}) \cap C(R^N)$;

2) $|\delta_r(x)| = r\,|x|$;

3) $|x^{-1}| = |x|$;

4) $|x| = 0$ iff $x = 0$.

There is a homogeneous norm $|\cdot|$ on $G$ and

$$\Gamma(x, y) = C_Q |x^{-1} \cdot y|^{2-Q} \tag{4}$$