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EXISTENCE AND UNIQUENESS OF RADIAL SOLUTIONS OF QUASILINEAR EQUATIONS IN A BALL*

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Abstract We consider the boundary value problem for the quasilinear equation

$$\mathrm{div}(A(|Du|)Du) + f(u) = 0,\ u > 0,\ x \in B_R(0),\ u|_{\partial B_R(0)} = 0,$$

where A and f are continuous functions in $(0, \infty)$ and f is positive in $(0, 1)$, $f(1) = 0$. We prove that (1) if f is strictly decreasing, the problem has a unique classical radial solution for any real number $R > 0$; (2) if f is not monotonous, the problem has at least one classical radial solution for some $R > 0$ large enough.
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1. Introduction

In this paper we consider the boundary value problem for the quasilinear equation

$$\begin{cases} \mathrm{div}(A(|Du|)Du) + f(u) = 0,\ u > 0, & x \in B_R(0), \\ u = 0, & x \in \partial B_R(0), \end{cases} \tag{1.1}$$

where $B_R = B_R(0)$ is a ball in $\mathbf{R}^n$ with radius R, $A(p)$ is a real, positive and continuous function defined for $p > 0$. Detailed conditions on A, f will be given later.

Our interesting is in the positive radial solutions of (1.1). In this way, the problem (1.1) is, in fact, the following problem

$$\begin{cases} (Au')' + \dfrac{n-1}{r}Au' + f(u) = 0, \\ u > 0,\ 0 < r < R, \\ u'(0) = 0,\ u(R) = 0, \end{cases} \tag{1.2}$$

where $A = A(|u'|)$, $u = u(r) = u(|x|)$.

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Definition *By a classical solution* u *of (1.2), we mean that* $u \in C^1([0, R])$, $A(|u'|)u' \in C^1([0, R])$ *and* u *satisfies (1.2).*

Our main technique is the shooting argument. Firstly, we consider the initial problem

$$(Au')' + \frac{n-1}{r}Au' + f(u) = 0,\ r > 0;\ u'(0) = 0,\ u(0) = \alpha \tag{1.3}$$

and we get a maximal existence interval $(0, R(\alpha))$ on which $u > 0$. Inversely, for a given $R > 0$, we find a suitable α such that $R = R(\alpha)$.

For the development of radial solutions of quasilinear equations, we refer the readers to [1] and the references therein, for shooting argument, to [1– 3].

Now we give the motivation of this paper. In [2], the ground states of quasilinear equations

$$\mathrm{div}(A(|Du|)Du) + f(u) = 0,\ x \in \mathbf{R}^n$$

is considered. A ground state is a positive radial solution such that $u \to 0$ as $|x| \to \infty$. In [2], the conditions on A are similar to ours, but it has two nearly necessary (just as pointed out in [3]) conditions on f: (a) there exists $\beta > 0$ such that $F(u) < 0$ for $0 < u < \beta$, where $F(u) = \int_0^u f(s)ds$; (b) there exists $\gamma > \beta$ such that $f(u) > 0$ for $\gamma > u \geq \beta$ and $f(\gamma) = 0$ if $\gamma < \infty$. The two conditions turn out that f must be negative in a neighborhood of zero. But our f is positive in $(0, 1)$. In [1], Moxun Tang considered the existence and uniqueness of the radial solution of the problem for the $m - Laplacian$ quasilinear equation

$$\begin{cases} \mathrm{div}((|Du|)^{m-2}Du) + f(u) = 0,\ x \in \mathbf{R}^n \\ u > 0,\ x \in R^n;\ u \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $1 < m < n$. This problem is a special case of the one considered in this paper. Although f in [1], different from [3], is positive near zero, yet an additional requirement on f is added that there exists $\eta \in (0, \gamma)$ such that $\Phi \leq 0$ on $(0, \eta)$, and $\Phi \geq 0$ on (η, γ), where $\Phi(u) = (\frac{F(u)}{f(u)})' - \frac{1}{m} + \frac{1}{n}$. The existence of such an η in [1] is as important as the existence of β in [2] but it is not easy to determine them in applications. They used this assumption to get a priori estimate so the shooting method can be effective. We noticed that the technique in [3] depends on n which is exactly the dimension of dimensional space. So, the technique in [3] cannot be used here directly.

Different from [1, 2] we do not assume the existence of the parameters β, γ. When f is decreasing, motivated by [4, 5], we get that the problem (1.1) has a unique classical radial solution for any positive real number $R > 0$; when f is not decreasing,motivated by [6, 7], using a method similar to the blow-up method, we obtained that (1.1) has a classical radial solution when radius R is large enough.

The following material is organized as follows: in Section 2 we consider the initial problem (1.3) and give some properties of the solution, especially the properties of the existence radius $R(\alpha)$ of the solution. Section 3 proves the existence and uniqueness