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HÖRMANDER'S INEQUALITY FOR ANISOTROPIC PSEUDO-DIFFERENTIAL OPERATORS

Fabio Nicola
(Dipartimento di Matematica, Università di Torino,
Via Carlo Alberto, 10, 10123 Torino, Italy.)
(E-mail: nicola@dm.unito.it)
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Abstract We prove a generalization of Hörmander's celebrated inequality for a class of pseudo-differential operators on foliated manifolds.
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1. Introduction

In this note we are concerned with the problem of establishing lower bounds for anisotropic pseudo-differential operators defined on certain foliated manifolds. Precisely, the symbol class we consider is defined in terms of the weight function $[\xi]_M = \sum_{j=1}^n |\xi_j|^{\frac{1}{M_j}}$, $\xi \in \mathbb{R}^n$, where $M = (M_1, \ldots, M_n)$ is a fixed n-tuple of integer numbers, see Definition 2.1 below. In other words, we are fixing a so-called quasi-homogeneous structure, cf. Melrose [1]. The corresponding class of pseudo-differential operators was studied, among others, by Hunt and Piriou [2],[3], Parenti [4], Parenti and Segala [5], Segala [6], Lascar [7], Rodino and Nicola [8], Nicola [9],[10] (see also Robert [11] for a global version in $\mathbb{R}^n$). Such operators are within the Beals-Fefferman calculus and within the Weyl-Hörmander calculus, so that products, symbolic calculus, construction of parametrices, etc. run as standard. The main feature is instead expressed by their invariance under suitable changes of variables. Was Parenti [4] who first specified the class $\mathcal{G}_M$ of allowed diffeomorphisms: they are those diffeomorphisms ϕ whose transposed Jacobian matrix $d\phi^t$ is in a particular subgroup $GL_M(n, \mathbb{R})$ of the group $GL(n, \mathbb{R})$ of real invertible $n \times n$ matrices, cf. the next Definition 2.8. So, we can transfer the definition of anisotropic operators on those manifolds X which have a $\mathcal{G}_M$ structure, see, e.g., Reinhart [12]. We stress that such manifolds are, in particular, foliated manifolds. In [8],[9], we considered a vector bundle $T^*_M X$ on such a manifold X in order to give an invariant meaning to the quasi-homogeneous principal symbol of an anisotropic classical operator, i.e. an operator whose symbol has an asymptotic expansion in quasi-homogeneous terms. The vector bundle $T^*_M X$ is, in a natural way, a foliated manifold. Furthermore, as we shall see, each leaf is canonically a symplectic manifold.

Let us now come to the aim of this note. Consider a classical, formally self-adjoint and properly supported anisotropic pseudo-differential operator A on X. We know that when the quasi-homogeneous principal symbol a_m is positive quasi-elliptic, the classical Gårding Inequality holds. If $a_m \geq 0$ only, Segala [6] proved the so-called Sharp Gårding Inequality

$$(Au, u) \geq -C_K \|u\|^2_{\frac{m}{2}-\frac{1}{2}} \quad \forall u \in \mathcal{C}_0^\infty(K), \tag{1.1}$$

for every compact subset $K \subset X$ (here $\|\cdot\|_s$ denotes the norm in weighted Sobolev spaces modelled on our operators). Of course, (1.1) can be seen as a particular case of the general Sharp Gårding Inequality due to Hörmander [13], Theorem 18.6.7, in the frame of the Weyl calculus.

So, one expects that for anisotropic operators with double characteristics the lower bound (*Hörmander's Inequality*)

$$(Au, u) \geq -C_K \|u\|^2_{\frac{m}{2}-1} \quad \forall u \in \mathcal{C}_0^\infty(K) \tag{1.2}$$

holds, under hypotheses on the principal and subprincipal symbol of A and geometric assumptions on the characteristic set $\Sigma \subset T^*_M X \setminus 0$.

Indeed, in the homogeneous case, i.e. $M_j = 1$ for all $j = 1, \cdots, n$, Hörmander [14] proved that, supposing

(i) $\Sigma := \{(x,\xi) \in T^*X \setminus 0 : a_m(x,\xi) = 0\}$ is a smooth sub-manifold of $T^*X \setminus 0$,

(ii) the canonical symplectic form σ has constant rank on Σ,

(iii) $a_m(x,\xi)$ vanishes exactly to second order on Σ,

the lower bound (1.2) (now with the usual Sobolev norms and A in Hörmander's classes $\Psi^m_{\mathrm{cl}}(X)$) is equivalent to

$$\begin{cases} a_m \geq 0 \quad \text{on } T^*_M X \setminus 0, \\ a^s_{m-1}(\rho) + \mathrm{Tr}^+ F_A(\rho) \geq 0 \quad \forall \rho \in \Sigma, \end{cases} \tag{1.3}$$

where a^s_{m-1} is the subprincipal symbol of A and $\mathrm{Tr}^+ F_A(\rho)$, for $\rho \in \Sigma$, is the positive trace of the fundamental matrix $F_A(\rho)$, defined by

$$\frac{1}{2}(\mathrm{Hess}\, a_m(\rho) v, v) = \sigma(v, F_A(\rho) v) \quad v \in T_\rho T^*X, \ \rho \in \Sigma.$$

Explicitly $\mathrm{Tr}^+ F_A(\rho) = \sum_{\mu>0} \mu$ with $i\mu$ in the spectrum of $F_A(\rho)$.

We observe that assumptions (i),(ii),(iii) play an essential role: without them one can only prove the equivalence of (1.3) and the weaker Melin Inequality:

For any $\epsilon > 0$, for any $\mu < (m-1)/2$ and any compact $K \subset X$, there exists $C_{\epsilon,\mu,K}$ such that

$$(Au, u) \geq -\epsilon \|u\|^2_{\frac{m-1}{2}} - C_{\epsilon,\mu,K} \|u\|^2_\mu \quad \forall u \in \mathcal{C}^\infty(K).$$