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SPIKE-LAYERED SOLUTIONS OF SINGULARLY PERTURBED QUASILINEAR DIRICHLET PROBLEMS ON BALL*

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Abstract We consider the singularly perturbed quasilinear Dirichlet problems of the form

$$\begin{cases} -\epsilon\Delta_p u = f(u) \text{ in } \Omega \\ u \geq 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \end{cases}$$

where $\Delta_p u = \text{div}(|Du|^{p-2}Du), p > 1, f$ is subcritical. $\epsilon > 0$ is a small parameter and Ω is a bounded smooth domain in R^N ($N \geq 2$). When $\Omega = B_1 = \{x; |x| < 1\}$ is the unit ball, we show that the least energy solution is radially symmetric, the solution is also unique and has a unique peak point at origin as $\epsilon \to 0$.

Key Words Quasilinear Dirichlet problem; peak point; unique.
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1. Introduction

In this paper we study the following singularly perturbed problem

$$\begin{cases} -\epsilon\Delta_p u = f(u) \text{ in } \Omega \\ u \geq 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$, $f(u) = g(u) - u^{p-1}$, Ω is a bounded smooth domain in R^N ($N \geq 2$). $\Delta_p u = \text{div}(|Du|^{p-2}Du), Du = (D_1 u, \cdots, D_N u), D_i u = \dfrac{\partial u}{\partial x_i}$, $\epsilon > 0$ is a parameter. The function $g : R \to R$ satisfies the following assumptions.

(g1) $g \in C^1(R)$, $g(t) \equiv 0$ for $t \leq 0$ and $g(t) = \bigcirc(t^\beta)$ as $t \to 0$ with $\beta > p - 1$.

(g2) $g(t) = \bigcirc(t^q)$ as $t \to +\infty$, where $p - 1 < q < \frac{Np}{N-p} - 1$ if $p < N$ and $p - 1 < q < \infty$ if $p \geq N$.

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(g3) $g(t)/t^{p-1}$ is strictly increasing for $t > 0$, and $\lim_{t\to+\infty} g(t)/t^{p-1} = +\infty$.

(g4) If $G(t) = \int_0^t g(s)ds$, then there exists a constant $\theta \in (0, 1/p)$ such that $G(t) \le \theta t g(t)$ for $t \ge 0$.

From (g1) and (g3), it should be observed that there exists a unique $\bar{t}$ satisfying $\bar{t}^{p-1} = g(\bar{t})$. To state the last condition, we need to consider the problem in R^N:

$$\begin{cases} -\Delta_p w = g(w) - w^{p-1} \text{ and } w > 0 \text{ in } R^N \\ w(0) = \max_{x\in R^N} w(x) \text{ and } w(x) \to 0 \quad \text{as} \quad |x| \to +\infty \end{cases} \tag{1.2}$$

(g5) The problem (1.2) has a unique radially symmetric solution, and it is nondegenerate.

We note that the function $g(t) = t^q$ satisfies assumptions (g1)-(g5) if $p - 1 < q < \frac{Np}{N-p} - 1$ (see Theorem 3 and its Corollary in [1] and Appedix C in [2] for detail).

The study of the solutions to the related equations has received considerable attention in recent years. The equation (1.1) with $p = 2$ is known as the stationary equation of the Keller-Segal system in chemotaxis (see [3] and the references therein). It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation, see [4] for more details.

We define an "energy" $J_\epsilon : W_0^{1,p}(\Omega) \to R$ associated with (1.1) by

$$J_\epsilon(u) = \frac{\epsilon}{p}\int_\Omega |Du|^p dx - \int_\Omega F(u)dx.$$

The well-known mountain-pass lemma due to Ambrosetti and Rabinowitz(see[5]) implies that

$$c_\epsilon = J_\epsilon(u_\epsilon) = \inf_{l\in\Gamma} \max_{s\in[0,1]} J_\epsilon(l(s))$$

is a positive critical value of J_ϵ, where Γ is the set of all continuous paths joining the origin and a fixed nonzero element $e \in W_0^{1,p}(\Omega)$ such that $e \ge 0$ and $J_\epsilon(e) = 0$. It turns out that c_ϵ is the least positive critical value (see Lemma 2.2 below). Hence a critical point u_ϵ of J_ϵ with critical value c_ϵ is called least-energy solution (or mountain-pass solution) of (1.1).

The corresponding problem for the case $p = 2$ and more general $f(u)$ has been studied in [2, 3, 6] (for the Neumann problem), Lin, Ni and Takagi showed for ϵ sufficiently small the least-energy solution has only one local maximum point x_ϵ and $x_\epsilon \in \partial\Omega$. Moreover, $H(x_\epsilon) \to \max_{x\in\partial\Omega} H(x)$ as $\epsilon \to 0$, where $H(x)$ is the mean curvature of x at $\partial\Omega$. In [7-9] (for the Dirichlet problem), Ni and Wei obtained that for ϵ sufficiently small, the least-energy solution u_ϵ has at most one local maximum and it is achieved at exactly one point $x_\epsilon \in \Omega$. More precisely, $u_\epsilon(\cdot + x_\epsilon) \to 0$ in $C^1_{loc}(\Omega - x_\epsilon \setminus \{0\})$, where $\Omega - x_\epsilon = \{x - x_\epsilon | x \in \Omega\}$,

$$d(x_\epsilon, \partial\Omega) \to \max_{x\in\Omega} d(x, \partial\Omega) \text{ as } \epsilon \to 0.$$