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INITIAL BOUNDARY VALUE PROBLEM FOR A DAMPED NONLINEAR HYPERBOLIC EQUATION *

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Abstract In the paper, the existence and uniqueness of the generalized global solution and the classical global solution of the initial boundary value problems for the nonlinear hyperbolic equation

$$u_{tt} + k_1 u_{xxxx} + k_2 u_{xxxxt} + g(u_{xx})_{xx} = f(x,t)$$

are proved by Galerkin method and the sufficient conditions of blow-up of solution in finite time are given.

Key Words Nonlinear hyperbolic equation, initial boundary value problem, global solution, blow-up of solution

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1. Introduction

In this work we devote to the following damped nonlinear hyperbolic equation

$$u_{tt} + k_1 u_{x^4} + k_2 u_{x^4 t} + g(u_{xx})_{xx} = f(x,t), \qquad x \in \Omega,\ t > 0 \tag{1.1}$$

with the initial boundary value conditions

$$u(0,t) = u(1,t) = 0, \qquad u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t > 0, \tag{1.2}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega} \tag{1.3}$$

or with

$$u_x(0,t) = u_x(1,t) = 0, \qquad u_{x^3}(0,t) = u_{x^3}(1,t) = 0, \qquad t > 0, \tag{1.4}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega} \tag{1.5}$$

or with

$$u(0,t) = u(1,t) = 0, \qquad u_x(0,t) = u_x(1,t) = 0, \qquad t > 0, \tag{1.6}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \overline{\Omega}, \tag{1.7}$$

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where $u(x,t)$ denotes an unknown function, k_1 and k_2 are two positive constants, $g(s)$ is a given nonlinear function, $f(x,t)$ is a given function, $\varphi(x)$ and $\psi(x)$ are given initial value functions which satisfy the continuous conditions:

$$\varphi_{x^{2k}}(0) = \varphi_{x^{2k}}(1) = \psi_{x^{2k}}(0) = \psi_{x^{2k}}(1) = 0, \qquad (k = 0, 1) \ in \ (1.3);$$
$$\varphi_{x^{2k+1}}(0) = \varphi_{x^{2k+1}}(1) = \psi_{x^{2k+1}}(0) = \psi_{x^{2k+1}}(1) = 0, \qquad (k = 0, 1) \ in \ (1.5)$$

and $\Omega = (0,1)$.

The equation (1.1) describes the motion for a class of nonlinear beam models with linear damping and general external time dependent forcing; for more physical interpretation of the equation (1.1) we refer to [1, 2].

The equation (1.1) and its multidimensional case have attracted much attention in recent years; for the well-posedness we refer to [3–5]. In [2] the authors have proved that the problem (1.1), (1.6), (1.7) has a unique global weak solution. In [1] the authors have been successful in proving the global existence of weak solutions for the multidimensional problem (1.1), (1.6), (1.7) by using a variational approach and the semigroup formulation. The energy decay of the mutidimensional problem (1.1), (1.6), (1.7) was given in [6].

In this paper, we are going to prove that the problem (1.1)-(1.3) or the problem (1.1), (1.4),(1.5) has a unique generalized global solution and a unique classical global solution by Galerkin method. We shall also show that the problem (1.1), (1.6), (1.7) has a unique generalized local solution. Finally, some sufficient conditions of blow-up of the solution for the problem (1.1), (1.6), (1.7) are given.

Throughout this paper, we use the following notations:$\|\cdot\|$, $\|\cdot\|_{Q_t}$, $\|\cdot\|_\infty$, $\|\cdot\|_{p(\Omega)}$ and $\|\cdot\|_{p(Q_t)}$ denote the norm of spaces $L^2(\Omega)$, $L^2(Q_t)$, $L^\infty(\Omega)$, $H^p(\Omega)$ and $H^p(Q_t)$, where $Q_t = \Omega \times (0,t)$ and $1 \le p < \infty$.

2. Global existence and uniqueness of solutions

In order to prove that the problem (1.1)-(1.3) has the generalized global solution and the classical global solution, we now introduce an orthonormal base in $L^2(\Omega)$. Let $\{y_i(x)\}$ be the orthonormal base in $L^2(\Omega)$ composed of the eigenvalue problem

$$y'' + \lambda y = 0, \qquad x \in \Omega,$$
$$y(0) = y(1) = 0$$

corresponding to eigenvalue $\lambda_i (i = 1, 2, \cdots)$, where "$'$" denotes the derivative. Let

$$u_N(x,t) = \sum_{i=1}^{N} \alpha_{Ni}(t) y_i(x) \tag{2.1}$$

be Galerkin approximate solution of the problem (1.1)-(1.3), where $\alpha_{Ni}(t)$ $(i = 1, 2, \cdots, N)$ are the undetermined functions, N is a natural number. Suppose that the initial value functions $\varphi(x)$ and $\psi(x)$ may be expressed