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LOCAL ESTIMATES OF SINGULAR SOLUTION TO GAUSSIAN CURVATURE EQUATION

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Abstract In this paper, we derive the local estimates of a singular solution near its singular set Z of the Gaussian curvature equation

$$\Delta u(x) + K(x)e^{u(x)} = 0 \quad \text{in } \Omega \setminus Z,$$

in the case that $K(x)$ may be zero on Z, where $\Omega \subset R^2$ is a bounded open domain, and Z is a set of finite points.

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1. Introduction

In this paper, we study the local behavior of a smooth solution u of the Gaussian curvature equation

$$\begin{cases} \Delta u + K(x)e^u = 0 & \text{in } \Omega \setminus Z \\ u \in C^2 & \text{in } \Omega \setminus Z \end{cases} \tag{1.1}$$

near the singular set Z, where Z is a set of finite points, and Ω is a bounded open domain in R^2.

Using the blow-up analysis, we get the following

Theorem 1.1 *Let u be a solution of*

$$\begin{cases} \triangle u(x) + K(x)e^{u(x)} = 0 & \text{in} & \Omega\backslash Z \\ \int_\Omega e^{u(x)}dx < \infty, u \in C^2 & \text{in} & \Omega\backslash Z. \end{cases} \tag{1.2}$$

Denote $\bar{\Omega}$ the closure of Ω. Assume that $Z = \{p_1, \cdots, p_l\} \subset \bar{\Omega}$, $K \in C^0(\bar{\Omega})$ and $K(p_i) \geq a > 0$ for $i = 1, \cdots, l$. Then for any compact subset $\tilde{\Omega}$ of Ω, there is a positive constant c such that

$$d(x, Z)e^{u(x)/2} \leq c$$

holds for all $x \in \tilde{\Omega}$, where $d(x, Z)$ denotes the distance between x and Z.

We also consider the case that K is zero on the singular set, and the method of blow-up analysis also works. As a result, we have the following:

Theorem 1.2 *Let u be a solution of* (1.2), *where $Z = \{p_1, \cdots, p_l\} \subset \bar{\Omega}$, $K \in C^0(\bar{\Omega})$ and K satisfies the following:*

$$K(p_i) = 0, i = 1, 2, \cdots, l. \text{ and } K(x) > 0 \text{ for } x \in \bar{\Omega}\backslash Z; \tag{1.3}$$

$$\lim_{d(x,Z)\to 0} \frac{K(x + o(d(x,Z)))}{K(x)} = 1, \quad \text{where } \frac{|o(d(x,Z))|}{d(x,Z)} \to 0 \text{ as } d(x,Z) \to 0; \tag{1.4}$$

$$\Lambda := \lim_{d(x,Z)\to 0} \sup_{y \in B_{d(x,Z)/2}(x)} \frac{K(x)}{K(y)} < +\infty. \tag{1.5}$$

Then for any compact subset $\tilde{\Omega}$ of Ω, there is a positive constant c such that

$$d(x,Z) K^{\frac{1}{2}}(x) e^{u(x)/2} \le c$$

for all $x \in \tilde{\Omega}$, where $d(x,Z)$ denotes the distance between x and Z.

We show that the method of moving planes, which was devised by Alexandrov and developed further by Serrin [1], Gidas-Ni-Nirenberg [2], Chen-Li [3], and Chen-Lin [4], still works for estimating singular solutions of some nonlinear elliptic equations in 2-dimensional case. Now we state another main result in this paper as follows:

Theorem 1.3 *Let u be a solution of*

$$\begin{cases} \triangle u(x) + K(x) e^{u(x)} = 0 & \text{in} \quad \Omega\backslash Z \\ u \in C^2(\Omega\backslash Z), & \text{and} \quad \limsup\limits_{d(x,Z)\to 0} \int_{B_{d(x,Z)/2}(x)} e^{u(y)} dy < +\infty. \end{cases} \tag{1.6}$$

Assume $K(x) \in C^\alpha(\Omega)$ with $0 < \alpha \le 1$, and $0 < a \le K \le b$, where a, b are two positive constants. Suppose in addition that $u(x) \to +\infty$ as $d(x,Z) \to 0$, and Z is a finite set of points. Then for any compact subset $\tilde{\Omega}$ of Ω, there is a positive constant c such that

$$d(x,Z) e^{u(x)/2} \le c$$

holds for $x \in \tilde{\Omega}$, where $d(x,Z)$ denotes the distance between x and Z.

For the case that $K(x)$ may attain zero on Z, we show an analogue of Theorem 1.3. To simplify notations, we assume $K(x) = |x|^{2N}$, where N is a positive integer. Then we obtain the following:

Theorem 1.4 *Let u be a solution of*

$$\begin{cases} \triangle u(x) + |x|^{2N} e^{u(x)} = 0 & \text{in} \quad \Omega\backslash\{0\} \\ u \in C^2(\Omega\backslash\{0\}), & \text{and} \quad \limsup\limits_{x\to 0} \int_{B_{|x|/2}(x)} |y|^{2N} e^{u(y)} dy < +\infty. \end{cases} \tag{1.7}$$

Suppose in addition that $\lim\limits_{x\to 0} u(x) = +\infty$. Then for any compact subset $\tilde{\Omega}$ of Ω, there is a positive constant c such that

$$e^{u(x)/2} \le c|x|^{-2N} \log\left(|x|^{-1}\right)$$