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SHORT COMMUNICATION SECTION

NONLINEAR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS–EXISTENCE *

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1. Introduction

This paper concerns the following Cauchy problem for the nonlinear Schröinger equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\left\{f(x,t)\Delta u + p\nabla f(x,t)\cdot\nabla u + k(x,t)|u|^2 u\right\}, \quad x \in M, \quad t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where M is either $\mathbb{R}^m$ or $\mathbb{T}^m$ (the flat m-dimensional torus), $i = \sqrt{-1}$, ∇ denotes the gradient, Δ is the Laplace-Beltrami operator on M, p is a fixed real constant, f and k are appropriately smooth real-valued functions on $M \times [0,\infty)$ and $u \in \mathbb{C}^n$.

We note that when $f(x,t) \equiv 1$ and $k(x,t) \equiv$ constant, (1) is just the ordinary (homogeneous) cubic nonlinear Schrödinger equation, which has been extensively studied, see [1, 2, 3] and references therein. When the functions f, k are independent of variable

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t (inhomogeneous), we have the following conservation laws which are not available yet for non-autonomous case:

$$\int_M |u(t)|^2 f^{p-1} dM \equiv \int_M |u_0|^2 f^{p-1} dM,$$
$$E(u(t)) \equiv E(u_0), \quad \forall t \in [0, T),$$

where $u(t) \in H^1(M)$ $(0 \leq t < T)$ is any solution to (1) and

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 f^p dM - \frac{1}{4} \int_M |\nabla u|^2 f^{p-1} k dM.$$

The authors review their recent results ([4]) on the existence of the solutions to the Cauchy problem (1) in non-autonomous case. So, the references of this paper are not intended to be complete. The reader is referred to those cited in our paper for further references.

Throughout this paper we will use the following notations: As usual, let $W^{s,q}(M)$ $(0 \leq s < \infty, 1 \leq q \leq \infty)$ denote Sobolev spaces, $H^s(M) = W^{s,2}(M)$, $H^\infty(M) = \cap_{s=0}^\infty H^s(M)$, and $[\lambda]$ the integral part of the positive number λ. $\|\cdot\|_{L^2}, \|\cdot\|_{H^s}$ denote the usual norms on $L^2(M)$ and $H^s(M)$ respectively.

2. Existence and Uniqueness

To make simplicity, we may assume $k(x,t) \equiv 1$. The results can be generalized with no difficulty to general case by posting suitable conditions on k. We will be referring to the following assumptions:

(A1) There exists a positive function $L(t) \in C(\mathbb{R})$ such that

$$\inf_{x \in M} |f(x,t)| \geq L(t), \quad \text{for all } x \in M,\ 0 \leq t < \infty;$$

(A2) f is C^1 with respect to t and there exists a positive function $U(t) \in C(\mathbb{R})$ such that

$$\|\partial_t f(\cdot, t)\|_{L^\infty} \leq U(t), \quad \text{for all } 0 \leq t < \infty;$$

(A3) $(p-1)\partial_t f \leq 0$ and there exists a positive constant c such that

$$\|f^{-p}(\cdot,t)\|_{L^\infty} \leq c \quad \text{and} \quad \sup_{x \in M} |f(x,t)| \leq c \inf_{x \in M} |f(x,t)|, \quad \text{for all } 0 \leq t < \infty.$$

We make use of the following uniformly parabolic systems

$$\begin{cases} \dfrac{\partial u}{\partial t} = (i+\epsilon)\left\{\operatorname{div}(f(x,t)\nabla u)\right\} \\ \qquad\qquad +i\left\{(p-1)\nabla f(x,t)\cdot\nabla u + k(x,t)|u|^2 u\right\}, \quad x \in M,\ t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

to approximate our equation. By an approximation argument, we obtain the following results.