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BMO SPACES AND JOHN-NIRENBERG ESTIMATES FOR THE HEISENBERG GROUP TARGETS

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Abstract In this paper, the BMO spaces for the Heisenberg group targets are studied. Some properties of the BMO spaces and the John-Nirenberg estimates are obtained.

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1. Introduction

Let H^m denote Heisenberg group which is a Lie group that has algebra $\mathbf{g} = \mathbf{R}^{2m+1}$ with a nonabelian group law:

$$(x_u, y_u, t_u) \cdot (x_v, y_v, t_v) = (x_u + x_v, y_u + y_v, t_u + t_v + 2(x_u y_v - x_v y_u)), \tag{1}$$

for every $u = (x_u, y_u, t_u), v = (x_v, y_v, t_v) \in H^m$, where $x_u, y_u, x_v, y_v \in R^m, t_u, t_v \in R^1$. The Lie algebra is generated by the left invariant vector fields:

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, \cdots, m, \quad T = \frac{\partial}{\partial t}.$$

For every $u = (x_u, y_u, t_u), v = (x_v, y_v, t_v) \in H^m$, the metric $d(u, v)$ in Heisenberg group H^m is defined as ([1])

$$d(u,v)=|uv^{-1}|_{H^m}=\Big[\big((x_u - x_v)^2+(y_u - y_v)^2\big)^2 + (t_u - t_v+2(x_v y_u - x_u y_v)^2\Big]^{1/4}. \tag{2}$$

Let $\Omega \subset R^n (n \geq 2)$ be a bounded connected domain, α be a real number with $2 < \alpha < \infty$. In [2], L. Capogna and Fang-hua Lin introduced the characterizations of the Sobolev space $W^{1,\alpha}(\Omega, H^m)$. In this paper, we will study some properties of $BMO(\Omega, H^m)$ and give a John-Nirenberg inequality about the maps in $BMO(\Omega, H^m)$. The results are signification in further discussing the properties of $W^{1,\alpha}(\Omega, H^m)$ for $\alpha = 2$.

2. Preliminaries

Now we introduce some known results which are used in studying the properties of $BMO(\Omega, H^m)$.

Lemma 2.1 (C_p-inequality) *If $a_1, a_2, \cdots, a_n$ are nonnegative real numbers, then*

$$\left(\sum_{i=1}^{n} a_i\right)^p \le C_p \sum_{i=1}^{n} a_i^p,$$

where $C_p = 1$, if $0 < p < 1$ and $C_p = n^{p-1}$, if $p \ge 1$.

Lemma 2.2 ([3]) *For $u, v, w \in H^m$, we then have*

$$d(u, v) \le d(u, w) + d(v, w). \tag{3}$$

Lemma 2.3 (Vitali type covering Lemma) *Let μ be a Lebesgue measure on R^n and G be a family of closed balls with $\sup\{\dim B | B \in G\} < \infty$ that covers a set $A \subset R^n$, where $\mu(A) < \infty$. Then there is a countable disjoint subfamily G^* of G such that*

$$A \subseteq \bigcup_{B^* \in G^*} B^*, \quad \mu(A - \{\cup B^* | B^* \in G^*\}) = 0,$$

and for any $B \in G$, there is $B_k \in G^*$ such that $B \cap B_k \neq \phi$, and $B \subset B_k^*$, where B_k^* denotes the ball which is concentric with B_k and its radius is 5-times that of B_k.

Lemma 2.4 (Calderon-Zygmund type decomposition) *Let $B \subset R^n$ be a closed ball. Suppose $f : B \to [0, \infty)$ and $S > 0$ such that $⨍_B f(x)dx \le S$. Then there exists a sequence of mutually disjoint balls $B_1, B_2, \cdots$, $B_i \subset \Omega$, $i = 1, 2, \cdots$ such that that following conditions hold:*

(i) $f(x) \le S$, *a.e.* $x \in B \setminus \cup_i B_i^*$,

(ii) $S < ⨍_{B_i} f(x)dx \le 2^n S$, $i = 1, 2, \cdots$.

Where $⨍_B f(x)dx = \frac{1}{|B|}\int_B f(x)dx$ and $|B| = \mu(B)$ denotes the Lebesgue measure of B in R^n.

The proofs of Lemma 2.3 and Lemma 2.4 are found in [4].

Lemma 2.5 ([5]) *$BMO(R^n, R^1)$ is complete.*

3. $BMO(\Omega, H^m)$ and John-Nirenberg's Estimations

Definition 3.1 *Let $\Omega \subset R^n (n \ge 2)$ be a bounded connected domain and $1 < \alpha < \infty$. If $u(q) : \Omega \to H^m$ satisfies $\int_\Omega d(u(q), u(q_0))^\alpha dq = L < \infty$, then we say $u \in L^\alpha(\Omega, H^m)$; if*

$$\operatorname*{ess\,sup}_{p \in \Omega} d(u(q), u(q_0)) = L < \infty,$$