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CONTINUOUS DEPENDENCE FOR A BACKWARD PARABOLIC PROBLEM*

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Abstract We consider a backward parabolic problem arising in the description of the behavior of the toroidal part of the magenetic field in a dynamo problem. In our backward time problem, the media parameters are spatial distributed and the boundary conditions are of the Robin type. For this ill-posed problem, we prove that the solution depends continuously on the initial-time geometry.
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1. Introduction

Let $\Omega = (0,1)$. Consider the following backward parabolic problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial x}(b(x)u), & (x,t)\in \Omega\times(0,T) \\ -a(0)u_x(0,t) + hu(0,t) = 0, & t\in(0,T) \\ a(1)u_x(1,t) + Hu(1,t) = 0, & t\in(0,T) \\ u(x,0) = u_0(x) & x\in\Omega, \end{cases} \tag{1.1}$$

where h, H are known nonnegative constants. In this paper, we always assume that the smooth coefficients $a(x)$ and $b(x)$ satisfy

$$a(x) \ge a_0 > 0,\ x\in\Omega, \tag{1.2}$$

$$a(x), |a'(x)|, |b(x)|, |b'(x)|, |b''(x)| \le M_0,\ x\in\Omega \tag{1.3}$$

$$b(0) = b(1) = 0, \qquad b'(0), b'(1) \le 0 \tag{1.4}$$

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for two known positive constants a_0, M_0. In practice, the function $a(x)$ is the magnetic resistivity and physically $a(x)$ and $a'(x)$ are bounded. Furthermore, the velocity field $u(x,t)$ is described by (1.1), as is usual in the kinematic dynamo problem and thus, we may impose the bounds on $b(x)$ and its derivatives. So, (1.3) is not restrictive. As for (1.2), it guarantees that our problem is backward in time. (1.4) is assumed due to the technique reasons. In the cases $a(x) = 1$ and $b(x) = 0$, (1.2),(1.3) and (1.4) are satisfied automatically. Such a kind of problem appears not only in the kinematic dynamo problem, but also in the heat conduction problem([1],[2]).

It is well-known that (1.1) is ill-posed. Firstly, unlike the forward problem, (1.1) is not always solvable for any initial function $u_0(x)$. Secondly, even if there exists unique solution for some $u_0(x)$, the solution does not depend continuously on the initial data. Relatively speaking, the research on the ill-posedness caused by instability is much important from the practical point of view. That is, the initial data are given by measurements in practice which implies they are unavoidably polluted by some errors. In this case, it is important to know if the errors in the initial data have a pronounced effect on the solution.

If the initial data are given at the instant time $t = 0$ with some error level date $\delta > 0$, that is, we are given initial data $\hat{u}_0(x)$ at $t = 0$ satisfying

$$\|u_0 - \hat{u}_0\| \le \delta, \tag{1.5}$$

then we should estimate the error $u(x,t) - v(x,t)$ by $u_0(x) - \hat{u}_0(x)$, where $v(x,t)$ solves (1.1) with $u_0(x) = \hat{u}_0(x)$. There has been a long history for the researches on this topic. By the logarithmic convexity method, it is well-known that the stability for the solution in $t \in (0,T)$ can be restored if we give some a-priori bound to the solution at the final time $t = T$. In the case $a(x) = 1$ and $b(x) = 0$, such a result may be found in [3] for a backward problem with the Dirichilet boundary condition. In recent years, it is found that the conditional stability can be applied to treat the ill-posed problem by the regularization method([4],[5]).

However, there is another kind of error in the initial data in practice. That is, although the physical process is govern by (1.1), instead of giving the initial data $\hat{u}_0(x)$ at the same instant of time $t = 0$, we are given the initial data over a perturbation curve $t = \varepsilon f(x)$ for small $\varepsilon > 0$, where $f(x)$ satisfies

$$|f'(x)| \le F_0, \quad 0 \le f(x) \le 1, \quad f(0) = f(1) = 0. \tag{1.6}$$

In this case, although $v(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial v}{\partial x}\right) + \dfrac{\partial}{\partial x}(b(x)v), & (x,t) \in \Omega \times (0,T) \\ -a(0)v_x(0,t) + hv(0,t) = 0, & t \in (0,T) \\ a(1)v_x(1,t) + Hv(1,t) = 0, & t \in (0,T) \\ v(x,0) = \hat{u}_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$