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FORMATION OF SINGULARITIES FOR QUASILINEAR HYPERBOLIC SYSTEMS WITH CHARACTERISTICS WITH CONSTANT MULTIPLICITY

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Abstract In this paper we consider the Cauchy problem for quasilinear hyperbolic systems with characteristics with constant multiplicity. Without restriction on characteristics with constant multiplicity(> 1), under the assumptions that there is a genuinely nonlinear simple characteristic and the initial data possess certain decaying properties, the blow-up result is obtained for the C^1 solution to the Cauchy problem.
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1. Introduction and Main Result

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \qquad (\text{resp.} \quad A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.2}$$

We have

$$\det |l_{ij}(u)| \neq 0 \qquad (\text{equivallently,} \quad \det |r_{ij}(u)| \neq 0). \tag{1.3}$$

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{1.4}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{1.5}$$

where δ_{ij} stands for the Kronecker's symbol.

In the case that system (1.1) is strictly hyperbolic, namely, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u),$$

F.John[1] and T.P.Liu[2] have obtained the blow-up phenomenon of C^2 solution to the Cauchy problem of system (1.1) for initial data with compact support, provided that all characteristics are genuinely nonlinear or a non-empty part of characteristics is genuinely nonlinear, while another part of characteristics is linearly degenerate, respectively. By introducing the concept of weak linear degeneracy, Li Ta-tsien, Zhou Yi and Kong Dexing have given a complete result on the global existence and the blow-up phenomenon for the Cauchy problem of system (1.1) with small and decaying C^1 initial data in [3], [4] and [5]. Then, The results have been generalized to the non-strictly hyperbolic system with characteristics with constant multiplicity (see [6] and [7]), in which all characteristics with constant multiplicity (> 1) are linearly degenerate.

In this paper, we still consider the quasilinear hyperbolic system with characteristics with constant multiplicity, but there is no restriction on the linear degeneracy for characteristics with constant multiplicity(> 1). For system (1.1) with "small" C^1 initial data with certain decaying properties, we discuss the blow-up phenomenon of the C^1 solution.

For hyperbolic system (1.1) with characteristics with constant multiplicity, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)(i, j = 1, \cdots, n)$.

Without loss of generality, we suppose that, in a neighbourhood of $u = 0$,

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{1.6}$$

where $1 \le p \le n$. When $p = 1$, system (1.1) is strictly hyperbolic; while, when $p > 1$, system (1.1) is a non-strictly hyperbolic systems with characteristics with constant multiplicity $p(> 1)$.

For the Cauchy problem of system (1.1) with the following initial data

$$t = 0 : u = \varepsilon\psi(x), \tag{1.7}$$

where $\varepsilon > 0$ is a small parameter and $\psi(x) \in C^1$ satisfies

$$\sup_{x\in\mathbb{R}} \big\{(1 + |x|)(|\psi(x)| + |\psi'(x)|)\big\} < \infty, \tag{1.8}$$

the main result in this paper is

Theorem 1.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, system (1.1) is hyperbolic and (1.6) holds. Suppose furthermore that in the neighbourhood of $u = 0$, there exists $m \in \{p + 1, \cdots, n\}$ such that $\lambda_m(u)$ is genuinely nonlinear:*

$$\nabla\lambda_m(u)r_m(u) \neq 0. \tag{1.9}$$