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EXISTENCE AND UNIQUENESS OF THE CAUCHY PROBLEM FOR A GENERALIZED NAVIER-STOKES EQUATIONS

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Abstract We consider the Cauchy problem for a generalized Navier-Stokes equations with hyperdissipation, with the initial data in L^p_σ. We follow the theme of [1] but with more complicated analysis on the symbol and obtain the existence and uniqueness results.

Key Words Cauchy problem; generalized Navier-Stokes equations; hyperdissipation; mild solution.

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1. Introduction

We consider the Generalized Navier-Stokes equations in whole space:

$$\begin{cases} \partial_t u + (-\Delta)^d u + (u\cdot\nabla)u + \nabla p = 0, & (x,t)\in \mathbf{R}^n\times[0,T), \\ \text{div } u = 0, & (x,t)\in \mathbf{R}^n\times[0,T), \\ u|_{t=0} = a \in L^p_\sigma, \end{cases} \tag{1}$$

where d is a positive real number, $n \geq 2$, and $(-\Delta)^d$ is defined by $|\xi|^{2d}\hat{f}(\xi) = (\widehat{(-\Delta)^d f})(\xi)$. When $d = 1$, (1) is the normal Navier-Stokes equations and has been studied by many authors. When $n = 2$ and $d \geq 1$, (1) was studied by S. Tourville ([2]) and was referred as the Navier-Stokes equations with hyperdissipation. [2] also explained the derivation of the equations on the numerical aspects. See [3], [4], [5] and [6] for more detail. In this paper we follow the theme of [1] and give the existence and uniqueness of (1) for $d > 1/2$. The basic idea is the following. One firstly transforms (1) into mild integral equations, then gives a bilinear estimates, finally by the common iterative arguments, obtains the results. The key to this paper is Theorem 2.2. When d is any positive integer, it is easy to prove Theorem 2.2, because in the case, $\hat{\Omega}(\xi) = \mathrm{e}^{-t|\xi|^{2d}} \in \mathcal{S}$, the Schwartz class, so does $\Omega(x)$ and decreases rapidly in infinity. But when d is not an integer, $\hat{\Omega}(\xi) \notin \mathcal{S}$, neither Ω, then we cannot conclude directly that Ω decreases in a given order we need in infinity, so we need some more complicated analysis to obtain Theorem 2.2. Our results contain the results listed in [1]

This paper is organized in the following way. In Section 2, we will give some notations and basic but important propositions. Section 3 contains a bilinear estimate. Finally, in Section 4, we give the main theorems on the existence and uniqueness of (1) without proofs.

2. Preliminary

Now we give some notations and conventions. Let

$$C_{0,\sigma}^{\infty}(\mathbf{R}^n)=\{f(x)\in C_0^{\infty}(\mathbf{R}^n):\mathrm{div}f=0\},$$

and $\|\cdot\|_p$ be the norm of $L^p(\mathbf{R}^n)$, $1\le p\le\infty$. Let L^p_σ be the completion of $C_{0,\sigma}^{\infty}(\mathbf{R}^n)$ in L^p. Let I be an interval in $\mathbf{R}^+$ and $L^q(I,L^p(\mathbf{R}^n))$ be the Lebesgue space with mixed norm

$$||f||_{L^q(I,L^p(\mathbf{R}^n))}=\sum_{j=1}^{n}\left(\int\left(\int|f_j(x,t)|^p\,dx\right)^{q/p}dt\right)^{1/q}<\infty.$$

For simplicity we denote the norm of $L^q(I,L^p(\mathbf{R}^n))$ by $\|\cdot\|_{p,q,I}$, $\|\cdot\|_{p,q,T}$ for $I=[0,T)$, and $\|\cdot\|_{p,q}$ for $I=\mathbf{R}^+$ and let $L^{p,q}(S_T)=L^q((0,T);L^p(\mathbf{R}^n))$. Let $\alpha=(\alpha_{1,\cdots,}\alpha_n)$ be a multiindex, $\partial^\alpha f=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}f$, $\partial^{(k)}f$ be any $\partial^\alpha f$ with $|\alpha|=k$, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, $\xi^{(k)}$ be any ξ^α with $|\alpha|=k$. Let $P_{a,k}$ be any expression with the form

$$\sum_{j=0}^{k}c_{(j)}\xi^{(j)}\,|\xi|^{a-k-j}\,, \tag{2}$$

where $c_{(j)}$ is a real number, a, k are integers and $k\ge 0$, the expression may be taken differently from line to line. It is easy to verify the following basic Lemma.

Lemma 2.1 *By the definition of $P_{a,k}$, we have*

$$\partial^{(1)}P_{a,k}=P_{a,k+1}, \tag{3}$$

$$P_{a,k}(\xi)\,|\xi|^{-2}\,\xi^{(1)}=P_{a,k+1}(\xi), \tag{4}$$

$$P_{a,k}+P_{a,k}=P_{a,k}, \tag{5}$$

$$P_{a,k}\times P_{b,l}=P_{a+b,k+l}. \tag{6}$$

Remark 2.1 In Lemma 2.1, we mean that "=" holds in these equalities if there exist $P_{a,k}$, $P_{b,l}$ and $P_{a,k+1}$ and so on with the form (2). Also in this paper we always have the following convention. If an expression with $P_{a,k}$ holds, we mean that there is some $P_{a,k}$ with the form of (2) such that the expression holds.