J. Partial Diff. Eqs. **17**(2004), 71–79
 Vol.17 No.1

EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS OF A DEGENERATE PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS*

Sun Fuqin and Wang Mingxin
(Department of Mathematics, Southeast University,
Nanjing 210018, China)
(E-mail: mxwang@seu.edu.cn)
(Received May. 21, 2003)

Abstract In this paper, we study the non-negative solutions to a degenerate parabolic system with nonlinear boundary conditions in the multi-dimensional case. By the upper and lower solutions method, we give the conditions on the existence and non-existence of global solutions.

Key Words Degenerate parabolic system; global solution; blow-up in finite time.
2000 MR Subject Classification 35K65, 35K45, 35K50, 35B33.
Chinese Library Classification O175.26.

1. Introduction and Main Results

Let constants $m > 1$ and p, $q > 0$, and let $R^N_+ = \{(x_1, x') \,|\, x_1 > 0,\ x' \in R^{N-1}\}$. In this paper we study the non-negative solutions to the following degenerate parabolic system with nonlinear boundary conditions in half space

$$\begin{cases} u_t = \Delta u^m, \quad v_t = \Delta v^m, & x \in R^N_+, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} = v^p, \quad -\dfrac{\partial v^m}{\partial x_1} = u^q, & x_1 = 0, \quad t > 0. \end{cases} \tag{1}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in R^N_+, \tag{2}$$

where the initial data $u_0(x)$ and $v_0(x)$ are non-negative C^1 functions and satisfy the compatibility conditions

$$-\frac{\partial u_0^m}{\partial x_1} = v_0^p, \quad -\frac{\partial v_0^m}{\partial x_1} = u_0^q, \qquad x_1 = 0.$$

*This work was supported by the PRC grant NSFC 10171088.

Moreover, they are compactly supported in R_+^N, and if they are nontrivial, then we assume that they satisfy $u_0(0) > 0$, $v_0(0) > 0$.

Since the pioneering work of Fujita in the 1960's, much work on the global existence and blow-up to the nonlinear parabolic problems has been done, see [1–5] and the references therein. The main aim of this paper is to discuss the global existence and finite time blow-up of solution to the problem (**??**) by constructing self-similar upper solution that exists globally and lower solution that blows up in finite time. This method has been used by many authors, see [**?**, **?**, **?**, **?**, **?**] and the references therein.

For the scalar equation

$$\begin{cases} u_t = \Delta u^m, & x \in R_+^N, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} = u^p, & x_1 = 0, \quad t > 0, \\ u(x,0) = u_0(x), & x \in R_+^N, \end{cases} \tag{3}$$

where $u_0(x)$ has the similar properties to the functions of (**??**). Huang et al [**?**] obtained

(i) If $p \le p_0 = (m+1)/2$, then all the solutions of the problem (**??**) are global;

(ii) If $p_0 < p < p_c = m + 1/N$, then all the nontrivial solutions of the problem (**??**) blow up in finite time;

(iii) If $p > p_c$, then the solution of the problem (**??**) exists globally for the small initial data u_0, while blows up in finite time for the large initial data u_0.

In the paper [**?**], Quiros and Rossi studied the Fujita type curves of the following problem on the half-line

$$\begin{cases} u_t = (u^m)_{yy}, \quad v_t = (v^n)_{yy}, & y > 0, \quad t > 0, \\ -(u^m)_y(0,t) = v^p(0,t), & t > 0, \\ -(v^n)_y(0,t) = u^q(0,t), & t > 0, \end{cases} \tag{4}$$

with m, $n > 1$ and p, $q > 0$.

Definition 1 *A pair of functions (u, v) is called an upper solution (lower solution) of (**??**) if it satisfies*

$$\begin{cases} u_t \ge (\le)\, \Delta u^m, \quad v_t \ge (\le)\, \Delta v^m, & x \in R_+^N, \quad t > 0, \\ -\dfrac{\partial u^m}{\partial x_1} \ge (\le)\, v^p, \quad -\dfrac{\partial v^m}{\partial x_1} \ge (\le)\, u^q, & x_1 = 0, \quad t > 0. \end{cases}$$

Proposition 1 *Let $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ be the upper and lower solutions of (**??**) respectively. If there exists a number $t_0 \ge 0$ such that*

$$\begin{cases} \underline{u}(x,t_0) \le \bar{u}(x,t_0), \quad \underline{v}(x,t_0) \le \bar{v}(x,t_0), \quad x \in R_+^N, \\ \underline{u}(0,t_0) < \bar{u}(0,t_0), \quad \underline{v}(0,t_0) < \bar{v}(0,t_0), \end{cases}$$

then

$$\underline{u}(x,t) \le \bar{u}(x,t), \quad \underline{v}(x,t) \le \bar{v}(x,t),$$

as long as both pairs of functions exist.