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GLOBAL ATTRACTORS OF REACTION-DIFFUSION SYSTEMS AND THEIR HOMOGENIZATION*

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Abstract In this paper, we study the existence of the global attractor $\mathcal{A}^\varepsilon$ of reaction-diffusion equation

$$\partial_t u^\varepsilon(x,t) = A_\varepsilon u^\varepsilon(x,t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x,t)),$$

and the homogenized attractor $\mathcal{A}^0$ of the corresponding homogenized equation, then give explicit estimates for the distance between the attractor $\mathcal{A}^\varepsilon$ and the homogenized attractor $\mathcal{A}^0$.
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1. Introduction and Main Results

We consider the reaction-diffusion system

$$\begin{cases} \partial_t u^\varepsilon(x,t) = A_\varepsilon u^\varepsilon(x,t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x,t)), & (x,t) \in \Omega \times \mathbf{R}^+, \\ u^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad u^\varepsilon(x,t)|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^3$ and $0 < \varepsilon \le \varepsilon_0 < 1$. Here $u^\varepsilon = u^\varepsilon(x,t) = (u^1_\varepsilon, \cdots, u^k_\varepsilon)$ is an unknown vector-valued function. The second order elliptic differential operators A_ε have the form as follows:

$$A_\varepsilon u := \mathrm{diag}(A^1_\varepsilon u^1, \cdots, A^k_\varepsilon u^k), \tag{1.2}$$

with

$$A^l_\varepsilon u^l = \sum_{i,j=1}^{3} \partial_{x_i}(a^l_{ij}(\varepsilon^{-1}x)\partial_{x_j} u^l(x)), \tag{1.3}$$

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where the functions $a_{ij}^l(y)$, $l = 1, \cdots, k$, $y \in \mathbf{R}^3$, are assumed to be symmetric, smooth and $\mathbf{Y}$-periodic with respect to $y \in \mathbf{R}^3$, where $\mathbf{Y} \subset \mathbf{R}^3$ is a fixed cube. The uniform ellipticity condition

$$\sum_{i,j=1}^{3} a_{ij}^l(y)\zeta_i\zeta_j \geq \nu|\zeta|^2, \quad \forall y, \zeta \in \mathbf{R}^3, \tag{1.4}$$

is also assumed (with an appropriate $\nu > 0$) to be valid for operators A_ε^l. We impose that $f(x, y, u)$ is almost-periodic ([1]) with respect to $y \in \mathbf{R}^3$ and satisfies the conditions as follows:

$$f \in C^1(\mathbf{R}^k, \mathbf{R}^k), \quad \partial_z f(x, y, z)\zeta\zeta \geq -C_2\zeta\zeta, \quad \forall \zeta \in \mathbf{R}^k, \tag{1.5}$$

$$|f(x, y, u)| \leq C(1 + |u|^p), \quad \forall (x, y) \in \Omega \times \mathbf{R}^3, \tag{1.6}$$

$$\sum_{l=1}^{k} f^l u^l |u^l|^{p_l} \geq C \sum_{l=1}^{k} |u^l|^{p_l+2} - C_1, \quad \forall u \in \mathbf{R}^k, \tag{1.7}$$

where $p \geq 1, p_i \geq 2(p - 1)$, $i = 1, \cdots, k$. It is assumed also that the initial data $u_0 \in (L^2(\Omega))^k$.

Efendiev and Zelik (see [2]) studied the problem (1.1) when $f(x, y, u)$ is independent of y. Fiedler and Vishik (see [3]) studied the case when the $A_\varepsilon u$ in (1.1) is replaced by $a\Delta u$. In fact, one can obtain the existence of solutions and attractors for (1.1) with $f(x, y, u)$ depending on y by the standard method as those in [4]. However, when estimate the distance between the attractors for (1.1) and the attractors of the homogenized equation, the arguments in [2] or [3] don't work. We have to overcome these difficulties by combining the ideas in [3], [2] and analyzing carefully the properties of periodic and almost-periodic functions.

In order to simplify our expression, we denote $H = (L^2(\Omega))^k$, $V = (W_0^{1,2}(\Omega))^k$, $F = (L^\infty(\Omega))^k$, $\|\cdot\|_{(W^{l,p}(\Omega))^k} = \|\cdot\|_{l,p}$.

Theorem 1.1 *If the assumptions* (1.2) $-$ (1.7) *hold, and the initial data* $u_0 \in H$, *then for any* $T > 0$, $\varepsilon > 0$, *the problem* (1.1) *possesses a unique solution* $u^\varepsilon(x, t) \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$, $u^\varepsilon \in C(R^+; H)$. *The mapping* S_t^ε: $u_0 \longrightarrow u^\varepsilon(x, t)$ *defines a continuous semigroup* S_t^ε: $H \longrightarrow H$. *If, furthermore,* $u_0 \in V$, *then* $u^\varepsilon(x, t) \in L^\infty([0, T]; V) \cap L^2([0, T]; W^{2,2}(\Omega))$, $u^\varepsilon \in C(R^+; V)$.

Theorem 1.2 *If the assumptions* (1.2) $-$ (1.7) *hold, and* $u_0 \in H$, *then for every* $\varepsilon > 0$, *the semigroup* S_t^ε *generated by the equation* (1.1) *possesses a global compact attractor* $\mathcal{A}^\varepsilon$ *in* H.

Theorem 1.1 can be proved by the Faedo-Galerkin method with the help of R.Temam [4], and the details of the proof are omitted. Similar arguments as in [4] for the problem (1.1) yield the a prior estimates needed about $u^\varepsilon(x, t)$ in H and V, and we omit the details. Then Theorem 1.2, whose proof is also omitted, can be easily proved by the standard arguments [4, Theorem 1.1.1].

By the standard homogenization theory, one can obtain the homogenized problem (2.11), for which one can prove the similar results to Theorems 1.1 and 1.2. In order to