J. Partial Diff. Eqs. **17**(2004), 97–121
©International Academic Publishers Vol.17, No.2

GLOBAL WELL-POSEDNESS FOR THE KLEIN–GORDON EQUATION BELOW THE ENERGY NORM*

Miao Changxing
(Institute of Applied Physics and Computational Mathematics
PO Box 8009, Beijing 100088, China)
(E-mail: miao_changxing@mail.iapcm.ac.cn)
Zhang Bo
(School of Mathematical and Information Sciences, Coventry University
Coventry CV1 5FB, UK)
(E-mail: b.zhang@coventry.ac.uk)
Fang Daoyuan
(Department of Mathematics, Zhejiang University
Hangzhou, 310027, China)
(E-mail: dyfang@mail.hz.zj.cn)
(Received Feb. 11, 2004)

Abstract We study global well-posedness below the energy norm of the Cauchy problem for the Klein-Gordon equation in $\mathbf{R}^n$ with $n \geq 3$. By means of Bourgain's method along with the endpoint Strichartz estimates of Keel and Tao, we prove the H^s-global well-posedness with $s < 1$ of the Cauchy problem for the Klein-Gordon equation. This we do by establishing a series of nonlinear a priori estimates in the setting of Besov spaces.
Key Words Klein-Gordon equations; Strichartz estimates; Besov spaces; well-posedness.
2000 MR Subject Classification 35L05, 35L15.
Chinese Library Classification O175.27, O175.29.

1. Introduction and the Main Result

Recently, a large amount of work has been devoted to the study of the Cauchy problem for the semilinear wave equation

$$u_{tt} - \triangle u = -|u|^{\rho-1}u, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^n, \quad n \geq 3, \tag{1.1}$$

*The first author was partly supported by the NSF, Special Funds for Major State Basic Research Projects of China and NSF of China Academy of Engineering Physics.

$$u(x,0)=\phi(x), \quad x\in \mathbf{R}^n, \tag{1.2}$$

$$u_t(x,0)=\psi(x), \quad x\in \mathbf{R}^n \tag{1.3}$$

in energy space, that is, for initial data $(\phi,\psi)\in H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$, where $\rho>1$ and $H^s(\mathbf{R}^n)=(1-\Delta)^{-s/2}L^2(\mathbf{R}^n)$ for $s\in\mathbf{R}$. For example, the global well-posedness, the scattering theory as well as regularity of solutions to the Cauchy problem (1.1)-(1.3) have been established in, e.g. [1, 2], [3-7],[8, 9] and [10-12] for the case of sub-critical growth $1<\rho<1+4/(n-2)$ or in, e.g. [7, 13] and [9, 14, 15] for the case of critical growth $\rho=1+4/(n-2)$. However, certain questions remain open. For example, when $\rho>1+4/(n-2)$, it is not yet clear whether or not there exists a global, regular, solution to the Cauchy problem (1.1)-(1.3) with arbitrary initial data.

On the other hand, the local well-posedeness (as well as global well-posedness with small initial data in the critical growth case) in fractional Sobolev spaces has also been studied recently by many authors for the Cauchy problem of general semi-linear wave equations including (1.1)-(1.3) under minimal regularity assumptions on the initial data (see, e.g. [16], [17-20] and [9, 21, 22].However, very few authors have undertaken a study of global well-posedness below the energy norm of Cauchy problems with less regular initial data. In [23, 24], Bourgain established the global well-posedness of the Cauchy problem of nonlinear wave or dispersive wave equations for rough initial data (with infinite energy) for the first time. Bourgain's method has been further developed to prove the global well-posedness below the energy norm of the Cauchy problem for the modified KdV equation in [25] and for the semi-linear wave equation (1.1)-(1.3) under minimal regularity assumptions on the data in the three-dimensional case in [26]. It should be pointed out that Keel and Tao have recently proposed a different approach to the study of both local and global well-posedness below the energy norm for the wave map equation [27, 28] and the Yang-Mills equation [29].

In this paper we consider the following Cauchy problem for the Klein–Gordon equation:

$$u_{tt}-\triangle u+m^2u=-|u|^{\rho-1}u, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^n, \quad n\geq 3, \;\; m\neq 0, \tag{1.4}$$

$$u(x,0)=\phi(x), \quad x\in\mathbf{R}^n, \tag{1.5}$$

$$u_t(x,0)=\psi(x), \quad x\in\mathbf{R}^n, \tag{1.6}$$

and establish the global well-posedness below the energy norm for the Cauchy problem in the case of general spatial dimensions.

To deal with the case of general spatial dimensions, we have to develop some a priori nonlinear estimates in Besov spaces. Our purpose is to give a unified method to deal