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LOCAL WELL-POSEDNESS OF INTERACTION EQUATIONS FOR SHORT AND LONG DISPERSIVE WAVES*

Huo Zhaohui
(Graduate School, China Academy of Engineering Physics,
P.O. Box 2101, Beijing, 100088, China)
(E-mail: huozhaohui@yahoo.com.cn)
Guo Boling
(Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing, 100088, China)
(E-mail: gbl@mail.iapcm.ac.cn)
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Abstract The well-posedness of the Cauchy problem for the system

$$\begin{cases} i\partial_t u + \partial_x^2 u = uv + |u|^2 u, \quad t, x \in \mathbb{R}, \\ \partial_t v + \partial_x \mathcal{H} \partial_x v = \partial_x |u|^2, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), \end{cases}$$

is considered. It is proved that there exists a unique local solution $(u(x,t), v(x,t)) \in C([0,T); H^s) \times C([0,T); H^{s-\frac{1}{2}})$ for any initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R}) (s \geq \frac{1}{4})$ and the solution depends continuously on the initial data.

Key Words Short and long dispersive waves; the Fourier restriction norm; the Smoothing effects.

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1. Introduction

In this paper, we consider the existence and uniqueness of the solution to the Cauchy problem of the following interaction equations for the coupled Schrödinger–Benjamin-Ono equations.

$$\begin{cases} i\partial_t u + \partial_x^2 u = uv + |u|^2 u, \quad t, x \in \mathbb{R}, \\ \partial_t v + \partial_x \mathcal{H} \partial_x v = \partial_x |u|^2, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), \end{cases} \tag{1}$$

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where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x) = p.v.\frac{1}{\pi}\int \frac{f(x-y)}{y}dy.$$

There exists an interaction phenomenon between long waves and short waves under a weakly coupled nonlinearity which has been studied in various physical situations, cf. [?, ?]. The short wave is usually described by the Schrödinger type equation, cf. [?, ?]. The long wave is described by some sort of wave equation accompanied with a dispersive term, we refer to [?, ?]. In this paper, the long wave is described by Benjamin-Ono equation.

From mathematical point of view, the study of well-posedness of the system with dispersive terms, such as (1), is to consider the smoothing effects and the relation of the nonlinear terms of each equation of the system. The smoothing effects are induced by the linear parts of each equation of the system, the relations include both the interaction degree of their nonlinearity and the particular structure of the nonlinear terms of the equations.

Recently, the local well-posedness for single dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces with negative indices. For example, the one-dimensional nonlinear Schrödinger equation with appropriate quadratic nonlinearity is known to be well-posed up to $H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}), \forall\varepsilon > 0$ [?].

In general, a coupled system like (1) is more difficult to handle than to solve the Cauchy problem of each of the equations in the same spaces.

Recently, Bekiranov-Ogawa-Ponce [?] has considered the following system

$$\begin{cases} i\partial_t u + \partial_x^2 u = uv + |u|^2 u, \quad t, x \in \mathbb{R}, \\ \partial_t v + \nu\partial_x \mathcal{H}\partial_x v = \partial_x |u|^2, \\ u(0,x) = u_0(x), v(0,x) = v_0(x). \end{cases} \tag{2}$$

They proved that the system is locally well-posed in the space $L^2(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$ for $|\nu| < 1$. Indeed, the smoothing effect of the dispersive term $\nu\partial_x\mathcal{H}\partial_x u$ does not play a significant role in the solvability of (2), and the special structure of the nonlinear terms gives us some advantage for well-posedness.

However, for the case $\nu = 1$ or -1, it seems that there is a type of cancellation for the characteristics of the linear parts in (2) and the system does not have sufficient smoothing effects to guarantee its well-posedness in weaker spaces. They expected to prove the well-posedness of system (2) in the regular case $s > 0$ for $|\nu| = 1$.

In this paper, we will prove that the Cauchy Problem (2) is locally well-posed in $H^s(s \geq \frac{1}{4})$ if $|\nu| = 1$ by using the Fourier restriction norm method and the contraction