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THE EXISTENCE AND THE NON-EXISTENCE OF GLOBAL SOLUTIONS OF A FREE BOUNDARY PROBLEM

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Abstract We study a free boundary problem of parabolic equations with a positive parameter τ included in the coefficient of the derivative with respect to the time variable t. This problem arises from some reaction-diffusion system. We prove that, if τ is large enough, the solution exists for $0 < t < +\infty$; while, if τ is small enough, the solution exists only in finite time.
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1. Introduction

In this paper we study the following free boundary problem of parabolic equations:

$$\frac{1}{\tau}v_t = v_{xx} - 2v + H(x - \phi(t)), \quad x \in (0,1),\ t > 0; \tag{1.1}$$

$$v_x(0,t) = 0 = v_x(1,t), \quad t > 0; \tag{1.2}$$

$$v(x,0) = v_0(x), \quad x \in [0,1]; \tag{1.3}$$

$$\frac{d\phi}{dt} = C(v(\phi(t),t)), \quad t > 0; \tag{1.4}$$

$$0 < \phi(t) < 1, \quad t > 0; \tag{1.5}$$

$$\phi(0) = \phi_0 \in (0,1), \tag{1.6}$$

where τ is a positive constant, $H(s)$ is the Heaviside function, $x = \phi(t)$ is the free boundary, and

$$C(v) = \frac{2v - \frac{1}{2}}{\sqrt{(\frac{3}{4} - v)(v + \frac{1}{4})}}. \tag{1.7}$$

(1.1)-(1.6) is derived from the reaction-diffusion system

$$\begin{cases} \epsilon u_t = \epsilon^2 u_{xx} + f(u,v), \\ \dfrac{1}{\tau} v_t = v_{xx} + g(u,v), \end{cases} \tag{1.8}$$

where $f(u,v) = H(u - \frac{1}{4}) - u - v$ and $g(u,v) = u - v$. As $\epsilon \to 0$, the function $u(x)$ tends to 0 or 1 almost everywhere, and the layer between the regions $\{x|u(x) < 1/4\}$ and $\{x|u(x) > 1/4\}$ tends to an interface $x = \phi(t)$ which moves with the speed $C(v(t))$. The parameter τ in (1.8) is important because it represents the ratio of the dynamics of the interface and the bulk region. (For the backgrounds and derivations of (1.8) and (1.1)-(1.7), see [1, 2]).

D. Hilhorst, Y. Nishiura and M.Mimura [3] investigate the well-posedness of (1.1)-(1.6). They prove by a fixed-point argument that, if $v_0 \in L^2(0,1)$ and $-M \le v_0(x) \le M$ in $[0,1]$ for some suitable constant $M > 0$, then (1.1)-(1.6) has a unique weak solution $(v,\phi) \in L^2(0,T^*; H^1(0,1)) \times C^{0,1}([0,T^*])$ in the sense of distribution with $T^* > 0$ such that

$$T^* = +\infty, \quad \text{or,} \quad \lim_{t \to T^*-0} \phi(t) = 0 \text{ or } 1. \tag{1.9}$$

In the special case $v_0(x) = \frac{x}{2}$ and $\phi_0 = \frac{1}{2}$, one can easily verify that the free boundary of the unique solution of (1.1)-(1.6) is stationary such that $\phi(t) \equiv \frac{1}{2}$ for $0 \le t < +\infty$. When $v_0(x) = \frac{x}{2}$ but $\phi_0 \ne \frac{1}{2}$, numerical experiments (see [3, 4]) show that, if τ is large enough, the solution exists for $0 < t < +\infty$ and $\lim_{t \to +\infty} \phi(t) = \frac{1}{2}$; while, if τ is small enough, the solution exists only in finite time interval $[0,T^*]$ for some $T^* > 0$ and the free boundary $x = \phi(t)$ hits the boundary $x = 0$ or 1 as $t \to T^* - 0$. Moreover, for medium τ, the solution may exist for $0 \le t < +\infty$ and oscillate around $x = \frac{1}{2}$. YM. Lee, R. Schaaf and R. C. Thompson [4] studied this phenomenon in the view of the bifurcation theory and proved that there exists a critical $\tau_c > 0$ such that, as τ decreasingly crosses τ_c, the steady solution of (1.1)-(1.2) and (1.4)-(1.5) transfers from stable to unstable.

In this paper we shall rigorously prove that, if τ is large enough, the solution of (1.1)-(1.6) exists for $0 < t < +\infty$; while, if τ is small enough, the solution exists only in finite time. In order to give a precise statement of our results, we need some notations and assumptions.

Set

$$\begin{cases} v^-(x) = v(x), & \text{for } 0 \le x \le \phi(t),\ t \ge 0; \\ v^+(x) = v(x), & \text{for } \phi(t) \le x \le 1.\ t \ge 0. \end{cases} \tag{1.10}$$

Then, it is easy to verify that (1.1)-(1.6) is equivalent to the following free boundary problem:

$$\frac{1}{\tau} v_t^- = v_{xx}^- - 2v^-, \quad 0 < x < \phi(t),\ t > 0; \tag{1.11}$$

$$\frac{1}{\tau} v_t^+ = v_{xx}^+ - 2v^+ + 1, \quad \phi(t) < x < 1,\ t > 0; \tag{1.12}$$