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GLOBAL SOLUTION OF THE VLASOV-POISSON-LANDAU SYSTEMS NEAR MAXWELLIANS WITH SMALL AMPLITUDE

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Abstract Global-in-time classical solutions near Maxwellians with small amplitude are constructed for the Vlasov-Poisson system with certain generalized Landau collision operator. The construction of global solution is based on an energy method.
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1. Introduction

We consider the following system ([1])

$$\begin{aligned}&\partial_t F + v\cdot\nabla_x F + \nabla_x\phi\cdot\nabla_v F = Q[F,F],\\&\triangle\phi = \rho-\rho_0 = \int_{R^3} F dv - \rho_0, \qquad \int_{T^3}\phi dx = 0,\\&F(0,x,v) = F_0(x,v),\end{aligned} \tag{1.1}$$

where $F(t,x,v)$ is the spatially periodic distribution function for the particles at time $t\ge 0$, with spatial coordinates $x=(x_1,x_2,x_3)\in[-\pi,\pi]^3 = T^3$ and velocity $v=(v_1,v_2,v_3)\in R^3$. The collision between particles is given by Landau operator,

$$\begin{aligned}Q[F,G] &= \nabla_v\cdot\left\{\int_{R^3}\phi(v-v')[F(v')\nabla_v G(v) - G(v)\nabla_v F(v')]dv'\right\}\\&= \partial_i\int_{R^3}\phi^{ij}(v-v')[F(v')\partial_j G(v) - G(v)\partial_j F(v')]dv'.\end{aligned}$$

where $\phi^{ij} = \{\delta_{ij} - v_i v_j/|v|^2\}|v|^{\gamma+2}$. We are only concerned with $\gamma\ge -1$.

We study the classical solutions for (1.1) near a global Maxwellian $\mu = e^{-|v|^2}$ and $\rho_0 = \int_{R^3} e^{-|v|^2} dv$. We define the standard perturbation $f(t,x,v)$ to μ as $F = \mu + \mu^{1/2} f$. It is well known that $Q[\mu,\mu]=0$. By expanding $Q[\mu+\mu^{1/2}g_1, \mu+\mu^{1/2}g_2]$, we define

$$Q\left[\mu+\mu^{1/2}g_1, \mu+\mu^{1/2}g_2\right] \equiv Q[\mu,\mu] + \mu^{1/2}\left\{Kg_1 + Ag_2 + \Gamma[g_1,g_2]\right\}.$$

The system (1.1) for $f(t,x,v)$ turns into

$$[\partial_t + v\cdot\nabla_x + \nabla_x\phi\cdot\nabla_v]f - 2\nabla_x\phi\cdot v\mu^{1/2} + Lf = \nabla_x\phi\cdot vf + \Gamma[f,f],$$
$$\triangle\phi = \int_{R^3} f\mu^{1/2}dv, \quad \int_{T^3}\phi dx = 0, \quad f(0,x,v) = f_0(x,v), \tag{1.2}$$

where $L = -A - K$. Notice that A, K and Γ are defined in the same way as in [2], namely, $\sigma^{ij} = \phi^{ij} * \mu$,

$$Ag = \mu^{-1/2}\partial_i\left\{\mu^{1/2}\sigma^{ij}[\partial_j g + v_j g]\right\},$$
$$Kg = -\mu^{-1/2}\partial_i\left\{\mu\left[\phi^{ij} * \{\mu^{1/2}[\partial_j g + v_j g]\}\right]\right\},$$
$$\Gamma[g_1,g_2] = \partial_i\left[\left\{\phi^{ij} * [\mu^{1/2}g_1]\right\}\partial_j g_2\right] - \left\{\phi^{ij} * [v_i\mu^{1/2}g_1]\right\}\partial_j g_2$$
$$- \partial_i\left[\left\{\phi^{ij} * [\mu^{1/2}\partial_j g_1]\right\}g_2\right] + \left\{\phi^{ij} * [v_i\mu^{1/2}\partial_j g_1]\right\}g_2.$$

Obviously, the conservation of mass, momentum, and energy of (1.1) holds

$$\frac{d}{dt}\int\int F(t) = \frac{d}{dt}\int\int v_i F(t) = \frac{d}{dt}\Big\{\int\int |v|^2 F(t) + \int |\nabla_x\phi(t)|^2\Big\} = 0.$$

By assuming that initially $F_0(x,v)$ has the same mass, momentum and energy as Maxwellian μ, we can rewrite the conservation law as

$$\int\int f(t)\mu^{1/2} = \int\int v_i f(t)\mu^{1/2} = \Big\{\int\int |v|^2 f(t)\mu^{1/2} + \int |\nabla_x\phi(t)|^2\Big\} = 0.$$

We introduce a weight function of v as $\omega = \omega(v) = [1+|v|]^{\gamma+2}$. We denote the weighted L^2 norm as $|g|_{2,\theta}^2 = \int_{R^3}\omega^{2\theta}g^2 dv$, $\|g\|_\theta^2 = \int_{R^3\times T^3}\omega^{2\theta}g^2 dxdv$ where $\|\cdot\|_0 = \|\cdot\|$.

We define the weighted norm and the high order energy norm as

$$|g|_{\sigma,\theta}^2 = \int_{R^3}\omega^{2\theta}[\sigma^{ij}\partial_i g\partial_j g + \sigma^{ij}v_i v_j g^2]dv,$$
$$\|g\|_{\sigma,\theta}^2 = \int_{R^3\times T^3}\omega^{2\theta}[\sigma^{ij}\partial_i g\partial_j g + \sigma^{ij}v_i v_j g^2]dxdv,$$
$$E(f(t,x,v)) \equiv \sum_{|\alpha|+|\beta|\le N}[\frac{1}{2}\|\partial_x^\alpha\partial_v^\beta f(t)\|^2 + \int_0^t \|\partial_x^\alpha\partial_v^\beta f(s)\|_\sigma^2 ds],$$
$$E(f_0) = E(f(0)) \equiv \sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_v^\beta f_0\|^2,$$

where $|\cdot|_{\sigma,0} = |\cdot|_\sigma$, $\|\cdot\|_{\sigma,0} = \|\cdot\|_\sigma$ and $N \ge 8$.

In the following we give some lemmas without a proof which can be found in [2].

Lemma 1.1 *Let $|\beta| > 0$, $|\alpha| + |\beta| \le N$ and $\theta \ge 0$. Then for small $\eta > 0$, there exists $C > 0$ and $C_\eta = C_\eta(\theta) > 0$ such that*

$$-(\omega^{2\theta}\partial_v^\beta[Ag], \partial_v^\beta g) \ge |\partial_v^\beta g|_{\sigma,\theta}^2 - \eta\sum_{|\beta_1|\le|\beta|}|\partial_v^{\beta_1}g|_{\sigma,\theta}^2 - C_\eta|\mu g|_2^2, \tag{1.3}$$

$$|(\omega^{2\theta}\partial_v^\beta[Kg_1], \partial_v^\beta g_2)| \le \{\eta\sum_{|\beta_1|\le|\beta|}|\partial_v^{\beta_1}g_1|_{\sigma,\theta} + C_\eta|\mu g_1|_2\}|\partial_v^\beta g_2|_{\sigma,\theta}, \tag{1.4}$$