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THE AREA INTEGRAL ON NEGATIVE CURVATURE SPACE FORM

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Abstract In this note, we show that the area integral of positive harmonic functions on a constant negative curvature space form is almost everywhere finite with respect to a harmonic measure on $S(\infty)$.
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1. Introduction

In the study of harmonic functions in $\mathbb{R}_+^{n+1}$, the area integral [1]

$$S(u)(x^0)=\left(\iint_{\Gamma_\alpha^h(x^0)}|\nabla_0 u|^2 y^{1-n}\mathrm{d}y\mathrm{d}x\right)^{1/2},$$

plays an essential role, where

$$|\nabla_0 u|^2=\Big|\frac{\partial u}{\partial y}\Big|^2+\sum_{j=1}^n\Big|\frac{\partial u}{\partial x_j}\Big|^2,$$

$\nabla_0 u$ is the gradient with respect to the Euclidean metric, and

$$\Gamma_\alpha^h(x^0)=\Big\{(x,y)\in\mathbb{R}_+^{n+1}:|x-x^0|<\alpha y,\ 0<y<h\Big\}\quad x^0\in\mathbb{R}^n$$

is a nontangential cone.

If we use Poincaré metric

$$\mathrm{d}s_{-1}^2=\frac{\mathrm{d}y^2+\sum_{i=1}^n\mathrm{d}x_i^2}{y^2},\quad (y>0)$$

in $\mathbb{R}_+^{n+1}$, the volume element is $\mathrm{d}V_{-1} = y^{-n-1}\mathrm{d}x^1 \cdots \mathrm{d}x^n\mathrm{d}y$, and $|\nabla_{-1}u|^2 = y^2|\nabla_0 u|^2$, where ∇_{-1} is the gradient with respect to the Poincaré metric. So, one gets

$$|\nabla_{-1}u|^2\mathrm{d}V_{-1} = |\nabla_0 u|^2 y^{1-n}\mathrm{d}y\mathrm{d}x.$$

In $(\mathbb{R}_+^{n+1}, \mathrm{d}s_{-1}^2)$, $\gamma = \{(x, y) : \ x = x^0\}$ is a geodesic, so $|x - x^0| < \alpha y$ is equivalent to $d_{-1}((x, y), \gamma) < \alpha$. Therefore

$$\Gamma_\alpha(x^0) = \left\{(x, y) \in \mathbb{R}_+^{n+1} : d_{-1}((x, y), \gamma) < \alpha\right\},$$

and

$$S(u)(x^0) = \left(\int_{\Gamma_\alpha(x^0)} |\nabla_{-1}u|^2\mathrm{d}V_{-1}\right)^{1/2}.$$

Consequently we get a clear expression and explanation of the area integral by using Poincaré metric on $\mathbb{R}_+^{n+1}$.

A basic fact says, if u is a harmonic function in $\mathbb{R}_+^{n+1}$ with respect to the Euclidean metric, then $S(u)(x)$ is finite for almost all $x \in \mathbb{R}^n$. It is certainly interesting to see whether it is finite almost everywhere if u is a harmonic function with respect to the Poincaré metric. If $n = 2$, it is certainly right because harmonic functions are conformal invariant. In this paper we will show it is in true for $n > 2$.

In fact, we can define the area integral on any simply connected complete manifold M with non-positive sectional curvature. Recall that Aderson-Schoen [2] defined the non-tangential cone in M.

Definition A *Let $\gamma : \mathbb{R}^+ \to M$ be a geodesic ray in M asymptotic to $Q \in S(\infty)$ with $\gamma(0) = O$. Then a nontangential cone T_d at Q is a domain of the form*

$$T_d = T_d^o(Q) = \{x \in M : \rho(x, \gamma) < d\}.$$

We define the area integral on M as follows.

Definition 1 *If u is a smooth function defined on M, and $\xi \in S(\infty)$, then the area integral of u at ξ is defined by*

$$S(u)(\xi) = \int_{T_d^o(\xi)} |\nabla u|^2\mathrm{d}V. \tag{1}$$

In this paper, we mainly show that

Theorem *Let M be a simply connected, complete, n-dimentional Riemannian manifold with negative constant sectional curvature $K_M = -k^2, k > 0$. If u is a positive harmonic function on M, then $S(u)(\xi) < +\infty$ for almost all $\xi \in S(\infty)$ with respect to a harmonic measure on $S(\infty)$.*

We conjecture that the above result holds for any manifold with nonpositively sectional curvature.