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SHARP CRITERIONS OF GLOBAL EXISTENCE AND COLLAPSE FOR COUPLED NONLINEAR SCHRÖDINGER EQUATIONS*

Gan Zaihui
(College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066; Mathematical College, Sichuan University, Chengdu 610064, China)
(E-mail: tanliang2008cn@yahoo.com.cn)
Zhang Jian
(College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, China)
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Abstract In this paper, a series of sharp criterions of global existence and collapse for coupled nonlinear Schrödinger equations are derived out in terms of the characteristics of the ground state and the local theories. And the conclusion that how small the initial data are, the global solutions exist is proved.
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1. Introduction

In this paper, we consider the coupled nonlinear Schrödinger equations

$$\begin{cases} i\phi_t + p\Delta\phi = \phi(a_{11}|\phi|^2 + a_{12}|\psi|^2), & t \geq 0, \quad x \in R^N, \\ i\psi_t + q\Delta\psi = \psi(a_{21}|\phi|^2 + a_{22}|\psi|^2), & t \geq 0, \quad x \in R^N, \end{cases} \tag{1}$$

where (ϕ, ψ) is a pair of complex value functions of $(t, x) \in R^+ \times R^N$, Δ is the Laplace operator on R^N; p, q and $a_{jk}(j, k = 1, 2)$ are real parameters. (1) can be regarded as a model to describe the interaction of two waves in nonlinear optical media(see [1-3]).

Assume that in (1):

$$p < 0, \quad q < 0, \quad pa_{12} = qa_{21}, \tag{2}$$

and the scalar matrix

$$\begin{pmatrix} pa_{11} & pa_{12} \\ qa_{21} & qa_{22} \end{pmatrix} \text{ is negative definite.} \tag{3}$$

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In the case of the Cauchy problem of the single nonlinear Schrödinger equation

$$i\varphi_t + \Delta\varphi + \varphi|\varphi|^{\sigma-1} = 0, \quad t \geq 0, \quad x \in R^N, \quad \varphi(0,x) = \varphi_0(x), \quad x \in R^N, \qquad (4)$$

where $\varphi = \varphi(t,x): \quad R^+ \times R^N \to C$, and $\sigma > 1$, which may describe the propagation of a narrow electromagnetic beam through a nonlinear medium or electromagnetic wave in a plasma(see [4, 5]), there have been many investigations. Glassey[6], Tsutsumi[7], Ogawa and Tsutsumi[8, 9] researched the blowup properties of the solutions for (4); Ginibre and Velo[10, 11] studied the local and global existence of the solutions in the energy class for (4). Recently, on the basis of single evolution equation, some coupled nonlinear evolution equations are being investigated extensively(see [12-17])

In the study for (1), we are interested in studying the sharp sufficient conditions of global existence and collapse for the Cauchy problem of the equations (1) for $N = 2$ and 3. Especially, we are concerned with the relations between the global existence and collapse of the Cauchy problem of the equations (1) and the ground state, which is the positive solution of the nonlinear Euclidean scalar field equations

$$\begin{cases} \omega_1 u + p\Delta u = u(a_{11}|u|^2 + a_{12}|v|^2), \\ \omega_2 v + q\Delta v = v(a_{21}|u|^2 + a_{22}|v|^2). \end{cases} \qquad (5)$$

Here $(u,v) \in H^1(R^N) \times H^1(R^N), \quad \omega_1 > 0, \quad \omega_2 > 0$ are two real parameters. If pair of real functions $(u,v) = (u(x), v(x))$ verify (5) and

$$(u,v) \in H^1(R^N) \times H^1(R^N)/\{(0,0)\},$$

then $\phi(t,x) = e^{-i\omega_1 t}u(x), \quad \psi(t,x) = e^{-i\omega_2 t}v(x)$ verifies (1), which are a standing wave solutions of (1). In this paper, based on the existence of standing wave with the ground state[12], we derive out the sharp criterion for collapse and global existence by applying the potential well argument[18] and the concavity method[19]. And we show the conclusion: How small are the initial data, the global solutions of the Cauchy problem for (1) exist.

2. Basic Scheme

Consider the coupled nonlinear Euclidean scalar field equations (5) and

$$(u,v) \in H^1(R^N) \times H^1(R^N)/\{(0,0)\}. \qquad (6)$$

It is well known that for (5), there exists a unique positive symmetric solution $(D(x), Q(x))$, which is called the ground state.

When $N = 2$, for $(u,v) \in H^1(R^2) \times H^1(R^2)$, define two functionals as follows:

$$S_1(u,v) = \frac{1}{2}\int_{R^2} \left(p^2|\nabla u|^2 + q^2|\nabla v|^2\right) dx. \qquad (7)$$