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# THE BLOW UP LOCUS OF NONLINEAR ELLIPTIC EQUATIONS WITH SUPERCRITICAL EXPONENTS

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**Abstract** We consider the compactness theorem for the positive solutions of the equation

$$\Delta u + h_1 u^\alpha + h_2 u^\beta = 0 \text{ in } \Omega \subset \mathbf{R}^n$$

and obtain the measure estimate of the blow up set for positive smooth solutions $\{u_i\}$ of the above equation with $\{||u_i||_{H^1(\Omega)} + ||u_i||_{L^{\alpha+1}(\Omega)} + ||u_i||_{L^{\beta+1}(\Omega)}\}$ bounded.
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## 1. Introduction

Let $\Omega$ be an open subset of $\mathbf{R}^n$ $(n \geq 3)$. We consider the blow up locus of the solutions to the equation

$$\Delta u + h_1 u^\alpha + h_2 u^\beta = 0 \text{ in } \Omega \tag{1.1}$$

where $\alpha \geq \dfrac{n+2}{n-2}$, $\alpha + 1 \geq 2\beta > 2$, $h_i \in C^1(\Omega)\,(i = 1, 2)$; $a_i \leq h_i(x) \leq b_i$; $0 < a_i < b_i$ and $|\nabla \log h_i(x)| \leq C$ for $x \in \overline{\Omega}$ and $i = 1, 2$. Guo-Li [1] studied the equation in the case that $\beta = 1$.

We say that $u$ is a positive weak solution of (1.1) in $\Omega$ if $u \geq 0$ a.e. and if, for all $\phi \in C^\infty(\Omega)$ with compact support in $\Omega$,

$$-\int_\Omega u \Delta\phi dx = \int_\Omega [h_1(x) u^\alpha + h_2 u^\beta]\phi(x) dx. \tag{1.2}$$

We say that such a weak solution $u$ is stationary if, in addition, it satisfies

$$\int_\Omega \left[ \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} + \frac{1}{\alpha+1} u^{\alpha+1}\frac{\partial h_1}{\partial x_i}\phi^i + \frac{1}{\alpha+1} h_1 u^{\alpha+1}\frac{\partial \phi^i}{\partial x_i}\right.$$

$$+\frac{1}{\beta+1}u^{\beta+1}\frac{\partial h_2}{\partial x_i}\phi^i+\frac{1}{\beta+1}h_2u^{\beta+1}\frac{\partial \phi^i}{\partial x_i}\Big]\,dx=0 \tag{1.3}$$

for all regular vector field $\phi$ with compact support in $\Omega$ (summation over $i$ and $j$ is understood).

For the weak solutions in $H^1(\Omega)\cap L^{\alpha+1}(\Omega)\cap L^{\beta+1}(\Omega)$, this identity is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is

$$\frac{d}{dt}E(u_t)|_{t=0}=0,$$

where

$$E(u)=\frac{1}{2}\int_\Omega|\nabla u|^2dx-\frac{1}{\alpha+1}\int_\Omega h_1u^{\alpha+1}dx-\frac{1}{\beta+1}\int_\Omega h_2u^{\beta+1}dx$$

and $u_t(x)=u(x+t\phi(x))$. It is clear that a smooth solution is stationary.

Let $u\in H^1(\Omega)\cap L^{\alpha+1}(\Omega)\cap L^{\beta+1}(\Omega)$ be a positive solution of (1.1). We denote by $\Sigma$ the set of points $x\in\Omega$ such that $u$ is not bounded in any neighborhood $V$ of $x$ in $\Omega$. If $u$ is bounded in some neighborhood of $x$, then the classical regularity theory ensures that $u$ is regular in some neighborhood of $x$. Therefore $\Sigma$ is the singular set of $u$. In the paper [2], Pacard showed that the Hausdorff dimension of the singular set of a weak positive stationary solution $u$ of the equation $-\triangle u=u^\alpha$ in $\Omega$ is less than $n-2\frac{\alpha+1}{\alpha-1}$ if $\frac{n+2}{n-2}\le\alpha\le\frac{n+1}{n-3}$. Guo-Li [1] showed that the Hausdorff dimension of the blow up set of the equation $\Delta u+h_1u+h_2u^\alpha=0$ is less than $n-2\frac{\alpha+1}{\alpha-1}$, where $\alpha\ge\frac{n+2}{n-2}$, $h_i\in C^1(\Omega)(i=1,2)$; $a_i\le h_i(x)\le b_i$; $0<a_i<b_i$ and $|\bigtriangledown\log h_i(x)|\le C$ for $x\in\overline{\Omega}$ and $i=1,2$.

In this paper, we consider the compactness theorem for the positive solutions of the equation (1.1). We shall first establish a monotonicity inequality for the positive stationary weak solutions $u\in H^1(\Omega)\cap L^{\alpha+1}(\Omega)\cap L^{\beta+1}(\Omega)$ by the similar idea in [2]. Then using such monotonicity property of energy and the idea of Schoen [3], we obtain the measure estimate of the blow up set for positive smooth solutions $\{u_i\}$ of (1.1) with $\Big\{||u_i||_{H^1(\Omega)}+||u_i||_{L^{\alpha+1}(\Omega)}+||u_i||_{L^{\beta+1}(\Omega)}\Big\}$ bounded.

More precisely, we prove the following theorem

**Theorem 1.1** *Let $\alpha\ge\frac{n+2}{n-2}$, $\alpha+1\ge2\beta>2$. Let $\{u_i\}$ be a sequence of positive smooth solutions of (1.1) with $\Big\{||u_i||_{H^1(\Omega)}+||u_i||_{L^{\alpha+1}(\Omega)}+||u_i||_{L^{\beta+1}(\Omega)}\Big\}$ bounded. Let $u$ be the weak limit of $\{u_i\}$ in $H^1(\Omega)\cap L^{\alpha+1}(\Omega)\cap L^{\beta+1}(\Omega)$. Then there is a subsequence of $\{u_i\}$ which converges uniformly in $C^k$ norm to $u$ away a closed set $\Sigma$ of locally finite $\mu-$ dimensional Hausdorff measure, where $\mu=n-2\frac{\alpha+1}{\alpha-1}$. Moreover, $\Sigma$ is a rectifiable set.*