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MAXIMUM PRINCIPLES FOR SECOND-ORDER PARABOLIC EQUATIONS

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Abstract This paper is the parabolic counterpart of previous ones about elliptic operators in unbounded domains. Maximum principles for second-order linear parabolic equations are established showing a variant of the ABP-Krylov-Tso estimate, based on the extension of a technique introduced by Cabré, which in turn makes use of a lower bound for super-solutions due to Krylov and Safonov. The results imply the uniqueness for the Cauchy-Dirichlet problem in a large class of infinite cylindrical and non-cylindrical domains.
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1. Introduction and Statement of the Results

Maximum principles are basic tools in the study of both linear and nonlinear partial differential equations. In two recent papers [1] and [2] we stated maximum principles for second-order linear elliptic operators, extending the results of Cabré [3]. Here we are concerned with the parabolic operators

$$Lw := -\partial_t w + a_{ij}(x,t)\partial_{ij}w + b_i(x,t)\partial_i w + c(x,t)w \tag{1.1}$$

in a domain D of $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$, with coefficients

$$a_{ij} = a_{ji},\ b = (b_i b_i)^{1/2} \in L^\infty(D), \quad i,j = 1,\dots,n, \tag{1.2}$$

$$\gamma_0|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Gamma_0|\xi|^2 \quad \forall\xi \in \mathbb{R}^n,\ \gamma_0 > 0. \tag{1.3}$$

In this case the maximum principle, which is related to the uniqueness in the Cauchy-Dirichlet problem for parabolic equations, is formulated in a different manner with respect to the elliptic case: it says that the solutions are controlled by the values on the "parabolic boundary" of D in $\mathbb{R}^{n+1}$, which for instance, in the case of a cylindrical domain $D = \Omega \times (0,T)$, consists of the union of lower base $\Omega \times \{0\}$ and side surface $\partial\Omega \times [0,T]$. To treat more general domains, following Cabré [3], which in turn is based on Krylov [4], we define the parabolic boundary $\partial_p D$ of D as the set of all the points $(y,s) \in \partial D$ for which there exist $\varepsilon > 0$ and a continuous function $x(t),\ t \in [s, s+\varepsilon]$ such that

$$x(s) = y \quad \text{and} \quad (x(t),t) \in D \text{ for } t \in]s, s+\varepsilon]. \tag{1.4}$$

Let $W^{2,1}_{n+1,loc}(D)$ be the class of functions which belong to $W^{2,1}_{n+1}(H)$ for all bounded open subsets H of D, where $W^{2,1}_{n+1}(H)$ is the completion of $C^\infty(\bar{H})$ under the norm

$$\begin{aligned} \|w\|_{W^{2,1}_{n+1}(H)} = \|\partial_t w\|_{L^{n+1}(H)} &+ \sum_{i,j}^{n} \|\partial_{ij} w\|_{L^{n+1}(H)} \\ &+ \sum_{i}^{n} \|\partial_i w\|_{L^{n+1}(H)} + \|w\|_{C(\bar{H})}. \end{aligned} \tag{1.5}$$

Definition 1.1(maximum principle) *We say that the maximum principle holds for the operator L in D if*

$$\begin{cases} Lw \geq 0 & \text{in } D \\ w \leq 0 & \text{on } \partial_p D \end{cases} \tag{1.6}$$

implies $w \leq 0$ in D for $w \in W^{2,1}_{n+1,loc}(D) \cap C(\bar{D})$ bounded above.

We will assume $c(x,t) \leq 0$, but in the case of domains which are bounded from below in the time-direction, it is sufficient for the maximum principle where $c(x,t)$ is bounded above, due to the fact that for a subsolution $w(x,t)$ of $Lw = 0$, the function $v(x,t) = e^{-\lambda t} w(x,t)$ is a subsolution of $Lv - \lambda v = 0$. Moreover we observe that already in the elliptic case the maximum principle may fail to hold when w is not bounded above.

It is well known that the maximum principle holds in the upper half-space $\mathbb{R}^n \times \mathbb{R}_+$. This is based on the strong maximum principle due to Nirenberg [5], which asserts that, if the maximum $M = \sup_D w^+$ of a subsolution w of $Lw = 0$ in a domain D of $\mathbb{R}^{n+1}$ achieved in a point $(\bar{x},\bar{t}) \in D$, then $w = M$ in every point $(x,t) \in D$, which can be joined with $(\bar{x},\bar{t})$ through a path in D consisting only of horizontal segments and upwards vertical segments. The maximum principle follows by finding suitable barrier functions which allow to locate M at a finite point (see [6]). This seems not possible in general in the case of a domain which is not bounded from below in the time-direction. On the other part from the elliptic theory we deduce that the maximum principle is violated in the exterior domain outside the cylinder $B_1 \times \mathbb{R}$, over the unit ball B_1 in $\mathbb{R}^3$ centered at the origin, by the stationary solution $w(x,t) = 1 - 1/|x|$.

Our purpose is to show that however in a large class of domains D, which are not bounded from below in the time-direction, the maximum principle continues to hold. We will need, in a sense that will be clear below, ”enough parabolic boundary” near all the points of D. The idea is to adapt the device of Cabré [3] for the parabolic