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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS FOR SEMILINEAR HEAT EQUATION ON UNBOUNDED DOMAIN

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Abstract In this paper, we consider the existence and nonexistence of global solutions to the semilinear heat equation $u_t - \triangle u = u^p$ with Neumann boundary value $\dfrac{\partial u}{\partial \nu} = 0$ on some unbounded domains, where $p > 1, \nu$ is the outward normal vector on boundary $\partial\Omega$. We prove that there exists a critical exponent $p_c = p_c(\Omega) > 1$ such that if $p \in (1, p_c]$, for nonnegative and nontrivial initial data, all positive solutions blow up in finite time; if $p > p_c$, for suitably small nonnegative initial data, there exists a global positive solution.

Key Words Semilinear heat equation; global existence; critical exponent of Fujita's type.

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1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t - \triangle u = u^p, \qquad x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0 \not\equiv 0, \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 2)$ is an unbounded connected domain with smooth boundary $\partial\Omega$, ν is the outward normal vector on boundary $\partial\Omega$, $p > 1$ and the initial data $u_0(x) \geq 0$, for $x \in \bar{\Omega}$.

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It is well known that the solution of (1.1) will blow up in a finite time if the initial data is suitably large [1]. An interesting question is what happens for small initial data. For the following Cauchy problem,

$$\begin{cases} u_t - \triangle u = u^p, & x \in R^N, \quad t > 0, \quad p > 1, \\ u(x,0) = u_0(x) \geq 0 \not\equiv 0, & x \in R^N. \end{cases} \tag{1.2}$$

In 1966, Fujita ([2]) proved that there exists a critical exponent $p_c = 1 + \frac{2}{N}$ (see [3] for the case $p = p_c$) such that all positive solutions of (1.2) blow up in a finite time if the exponent $p \leq p_c$, while global solution exists if $p > p_c$ and initial value is small enough. Since then, people call such critical exponent is the exponent of Fujita type.. The reader can find an excellent survey(see [4]) on this object for the equations with nonlinear reactions.

For the heat equation with a nonlinear boundary condition, however, there has been not much progress on this object until very recent years. For the problem (1.1), in 2000, H.Levine and Q.S.Zhang (see [5]) proved that there also exists a Fujita's critical exponent $p_c = 1 + \frac{2}{N}(N \neq 2)$ when $\Omega \subset R^N$ is a complement of a bounded domain, but for $N = 2$, and $p > 2$, they did not prove the existence of global positive solution for small initial data. In this paper, we will solve this problem, this fills the gap of [5]. However, in [5] they only considered the exterior domain, but what are about the other unbounded domains such as unbounded convex cone and convex cylinder(which will be defined later)? Whether does there also exist the Fujita' critical exponent for the problem (1.1)? This is the main question we will discuss in this paper.

It should be pointed out that, in 1996, on several unbounded domains, B.Hu and H.M.Yin(see [6]) had discussed the following nonlinear boundary-value problem:

$$\begin{cases} u_t - \triangle u = 0, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = u^p, & x \in \partial\Omega, \quad t > 0, \quad p > 1, \\ u(x,0) = u_0(x) \geq 0 \not\equiv 0, & x \in \Omega. \end{cases} \tag{1.3}$$

They obtained some valuable results on the existence of Fujita's critical exponent for the problem (1.3). And by using the method in [6], it is not difficult to see that the local existence, comparison principle and uniqueness theorems hold for the following general problem:

$$\begin{cases} u_t - \triangle u = u^p, & x \in \Omega, \quad t > 0, \quad p > 1, \\ \dfrac{\partial u}{\partial \nu} = u^\alpha, & x \in \partial\Omega, \quad t > 0, \quad \alpha > 1, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.4}$$