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NON-NEGATIVE RADIAL SOLUTION FOR AN ELLIPTIC EQUATION

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Abstract We study the structure and behavior of non-negative radial solution for the following elliptic equation

$$\Delta u = u^{\nu}, \quad x \in \mathbb{R}^n$$

with $0 < \nu < 1$. We also obtain the detailed asymptotic expansion of u near infinity.
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1. Introduction

In this paper, we consider the structure and behavior of non-negative radial solution of the following nonlinear equation

$$\Delta u = u^{\nu}, \quad x \in \mathbb{R}^n, \quad 0 < \nu < 1. \tag{1.1}$$

Problem (1.1) appears in several applications in mechanics and physics, and in particular can be treated as the equation of equilibrium states in thin films. For backgrounds on (1.1), we refer to [1 - 7] and the references therein.

The Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n, \quad t > 0, \quad p > 1, \\ u|_{t=0} = \phi \in C_0(\mathbb{R}^n) \equiv C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n), \quad \phi \geq 0, \ \ \phi \not\equiv 0 \end{cases} \tag{1.2}$$

has been studied by many authors ([8 - 12]). The structure and expansion of the non-negative radial solution of the steady-state problem of (1.2) also have been studied in [12, 13]. The author also refers to [14, 15] when this paper is in preparation.

2. Structure and Behavior

Definition 2.1 *We say that u is a regular solution of (1.1) if $u \in C^2(\mathbb{R}^n)$ and u satisfies (1.1). We call u a singular solution of (1.1) if $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap C(\mathbb{R}^n)$ satisfies (1.1) in $\mathbb{R}^n \setminus \{0\}$ with non-removable zero at $x = 0$.*

In the follows, we set

$$\delta = \frac{2}{1-\nu}, \qquad L = \Big[\delta(\delta + n - 2)\Big]^{\frac{1}{\nu-1}}.$$

Proposition 2.2. *When $0 < \nu < 1$, all nontrivial non-negative radial regular solutions of (1.1) are included in a family $\{u_\alpha\}_{\alpha>0}$ with u_α being the unique positive solution of the problem*

$$\begin{cases} u'' + \dfrac{n-1}{r}u' = u^\nu \quad in \ \ (0,\infty), \\ u(0) = \alpha, \qquad u'(0) = 0. \end{cases} \tag{2.1}$$

u_α is increasing in r, $r^{2/(\nu-1)}u_\alpha(r) \to L$ as $r \to \infty$ and $u_\alpha(r) = \alpha u_1(\alpha^{(\nu-1)/2}r)$. Moreover, the only radial singular solution of (1.1) is

$$u_0(r) = Lr^{2/(1-\nu)}.$$

Proof We can obtain the result by phase plane analysis, see [16]. Assume u is a nontrivial non-negative radial solution of (1.1). Let

$$r = |x|, \quad t = -\ln r, \quad v(t) = r^{-\delta}u(r). \tag{2.2}$$

By u_r and u_{rr}, then we have

$$u_{rr} + \frac{n-1}{r}u_r = r^{\delta\nu}v^\nu,$$

i.e.,

$$v'' - (2\delta + n - 2)v' + (\delta^2 + n\delta - 2\delta)v = v^\nu. \tag{2.3}$$

Let $q(v) = v'(t)$, then $v_{tt} = \dfrac{dq}{dv}q$. Denote $C_0 = 2\delta + n - 2$ and by (2.3) we have

$$q\frac{dq}{dv} - C_0 q + v(L^{\nu-1} - v^{\nu-1}) = 0. \tag{2.4}$$

On (v, v_t) plane, we know $(L, 0)$ is the only unstable equilibrium point, which implies $v(t) \to L$ as $t \to -\infty$. That is

$$\lim_{r\to\infty} r^{-2/(1-\nu)}u(r) = L.$$

If $u_\alpha(0) = \alpha > 0$, by scaling invariance, we have $u_\alpha(r) = \alpha u_1(\alpha^{(\nu-1)/2}r)$. All solutions of (1.1) form a one-parameter family of solutions.