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LONG-TIME ASYMPTOTIC FOR THE DAMPED BOUSSINESQ EQUATION IN A CIRCLE

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Abstract The first initial-boundary value problem for the following equation

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \gamma\Delta(u^2)$$

in a unit circle is considered. The existence of strong solution is established in the space $C^0([0,\infty), H_r^s(0,1))$, $s < 7/2$, and the solutions are constructed in the form of series in the small parameter present in the initial conditions. For $5/2 < s < 7/2$, the uniqueness is proved. The long-time asymptotics is obtained in the explicit form.

Key Words Damped Boussinesq equation; initial-boundary value problem; long-time asymptotics.
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1. Introduction

The Boussinesq equation which was derived by Boussinesq [1] in 1872 governs the propagation of small amplitude, long waves on the surface of shallow water and possesses special, travelling-wave solutions called solitary waves which were discovered by Scott Russell [2] more than thirty years earlier. Boussinesq's theory was the first to give a satisfactory, scientific explanation of the phenomenon of the existence of solitary waves.

The classical Boussinesq equation can be written as follows

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.1}$$

where $u(x,t)$ is an elevation of the free surface of fluid, subscripts denote partial derivates, and the constant coefficients α and β depend on the depth of fluid and the characteristic speed of long waves. In the literatures [3–8] equation (1.1) was studied from various points of view. Clarkson [3] gave a general approach to constructing exact solutions. Hirota [4] deduced conservation laws and examined N-soliton interaction. In [7] Yajima investigated nonlinear evolution of a linearly stable solution. The Lax pair for the inverse scattering transform were constructed by Zakharov [8]. Nakamura [6] discovered the explode-decay solitary wave solutions of the "spherical"Boussinesq equation, and Hirota [5] applied the Wronskian technique for finding rational solutions of the "spherical"and classical Boussinesq equations.

Certain generalizations of equation (1.1) were considered in [9–11]

$$u_{tt} = -u_{xxxx} + u_{xx} + (f(u))_{xx}. \tag{1.2}$$

In [11] Tsutsumi and Matahashi proved local and global well-posedness of the Cauchy problem for equation (1.2) by means of transforming it into a system of nonlinear Schrödinger equation. In [9, 10] it was showed that certain solitary wave solutions of equation (1.2) were nonlinearly stable for a range of their wave speeds.

In [12–19] Varlamov considered the following damped Boussinesq equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \tag{1.3}$$

where the second term on the left-hand side is responsible for strong dissipation. Here $\beta \in R^1$, α and b are positive constants. In these papers, the systematic study of the long-time behavior of solutions of Cauchy problems, spatially periodic ones, and boundary-value problems for equation (1.3) were investigated, and long-time asymptotic was found in the explicit form. In [19] Varlamov considered equation (1.3) in a unit circle with radially symmetric initial data and homogeneous boundary conditions and rewrited the problem in polar coordinates, namely:

$$u_{tt} - 2b\Delta u_t = -\alpha\Delta^2 u + \Delta u + \beta\Delta(u^2), \tag{1.4}$$

where $\Delta = (1/r)\partial(r\partial r)$ is the radial part of the Laplace operator.

In the present paper we solve the first initial-boundary value problem for the following generalized damped Boussinesq equation

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t = \alpha\Delta^3 u - \beta\Delta^2 u + \Delta u + \gamma\Delta(u^2) \tag{1.5}$$

in a circle. The term $\alpha\Delta^3(\alpha \neq 0)$ represents the oscillations of and elastic surface in the presence of viscosity. Equation (1.4) possesses strong dissipation, and it is natural to expect the exponential decay in time. Indeed, we obtain the exponential decay of solutions for equation (1.4) under some assumptions on the initial -boundary data. The present work can also be regarded as the continuation of Varlamov [19].

The main tool for solving the problem in a circle is the Fourier-Bessel series. We construct the solution and obtain the long-time asymptotics in the explicit form.