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MINIMAL POSITIVE ENTIRE SOLUTION OF SEMILINEAR ELLIPTIC EQUATION

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Abstract In this paper, the singular semilinear elliptic equation

$$\Delta u + q(x)u^{\alpha} + p(x)u^{-\beta} - h(x)u^{\gamma} = 0,\ x \in \mathbf{R}^N,\ N \geq 3,$$

is studied via the super and sub-solution method, where Δ is the Laplacian operator, $\alpha \in [0, 1)$, $\beta > 0$, and $\gamma \geq 1$ are constants. Under a set of suitable assumptions on functions $q(x)$, $p(x)$ and $h(x)$, it is proved that there exists for the equation one and only one minimal positive entire solution.

Key Words Super and sub-solution method; minimal positive solution; singular semilinear elliptic equation.

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1. Introduction

In this paper, we consider the singular semilinear elliptic equation

$$\Delta u + q(x)u^{\alpha} + p(x)u^{-\beta} - h(x)u^{\gamma} = 0,\quad x \in \mathbf{R}^N,\ N \geq 3, \tag{1}$$

where Δ is the Laplacian operator, q, p and h are locally Hölder continuous functions, with exponent $\theta \in (0, 1)$, defined on $\mathbf{R}^N$; and $\alpha \in [0, 1)$, $\beta > 0$ and $\gamma \geq 1$ are constants.

We are concerned with the existence and uniqueness of minimal positive entire solutions of (1). By positive entire solution of (1) is meant a positive function $u \in C^{2,\theta}_{loc}(\mathbf{R}^N)$ which satisfies (1) at every point in $\mathbf{R}^N$. A positive entire solution of (1) is termed minimal if it is bounded above by a constant multiple of $|x|^{2-N}$ for all $|x| \geq 1$. This is due to the well-known fact that no positive solution of $\Delta u \leq 0$ in an exterior domain can decay more rapidly than a constant multiple of $|x|^{2-N}$, see [1].

In particular, as $p(x), h(x) = 0$ or $q(x), h(x) = 0$, (1) is the so-called generalized Emden-Fowler equation

$$\Delta u + K(x)u^{\lambda} = 0,\quad x \in \mathbf{R}^N, \tag{2}$$

where λ is a constant, and K is a positive locally $\theta-$ Hőlder continuous function in $\mathbf{R}^N$, for which the existence of positive entire solutions has been investigated under various hypotheses, see for references [2–8].

The equation (2) with $\lambda \in (0,1)$ is said to be of sublinear type; if $\lambda \in (1,+\infty)$, then (2) is said to be of superlinear type; and if $\lambda < 0$, then (2) is said to be of singular type. Such singular equations originally arise from the boundary layer theory of viscous fluids, see [9, 10] .

In [7], the author gave an open problem about the existence of minimal positive entire solution of the equation

$$\Delta u + q(x)u^{\alpha} + p(x)u^{-\beta} = 0, \quad x \in \mathbf{R}^N, \; N \geq 3. \tag{3}$$

Using the results of [11] we can solve this problem when $\alpha, \beta \in (0,1)$, and $p(x), q(x)$ are positive in $\mathbf{R}^N$.

We will give some new results regarding the existence and uniqueness of minimal positive entire solutions of (1), which is more general than (3) not only in type of nonlinear terms but also in positivity condition of coefficient functions. The main result and its proof are given in Section 2. For completeness, the proof of Lemma 2 is given in the appendix of this article.

Our work is based upon the following three lemmas.

Lemma 1 *Let $f(x,s)$ be a function defined on $\mathbf{R}^N \times (0,+\infty)$ which is locally Hölder continuous with exponent $\theta \in (0,1)$ and is uniformly locally Lipschitz continuous in s for all x in any compact subset of $\mathbf{R}^N$. Suppose that there exist functions $v, w \in C^2(\mathbf{R}^N)$ such that for all $x \in \mathbf{R}^N$,*

$$\Delta v(x) + f(x, v(x)) \leq 0, \tag{4}$$

$$\Delta w(x) + f(x, w(x)) \geq 0, \tag{5}$$

and

$$v(x) \geq w(x) > 0. \tag{6}$$

Then the equation

$$\Delta u + f(x,u) = 0 \tag{7}$$

has a positive entire solution u such that

$$w(x) \leq u(x) \leq v(x), \quad x \in \mathbf{R}^N. \tag{8}$$

For the proof of this lemma please refer to [8] or [12].

A function v [resp. w] in $C^2(\mathbf{R}^N)$ that satisfies (4) [resp.(5)] is called a supersolution [resp. subsolution] of the equation (7) in $\mathbf{R}^N$.

Lemma 2 *The equation*

$$\Delta w + q(x)w^{\alpha} + p(x)w^{-\beta} = 0, \quad x \in \mathbf{R}^N, \; N \geq 3 \tag{9}$$