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LIOUVILLE TYPE THEOREMS OF SEMILINEAR EQUATIONS WITH SQUARE SUM OF VECTOR FIELDS

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Abstract Let X_j, $j = 1, \cdots, k$, be first order smooth quasi-homogeneous vector fields on $\mathcal{R}^n$ with the property that the dimension of the Lie algebra generated by these vector fields is n at $x = 0$ and $X_j^* = -X_j$, $j = 1, \cdots, k$. Let $L = \sum_{i=1}^{k} X_i^2$. In this paper, we study the nonnegative solutions of semilinear equation

$$Lu + f(x, u) = 0 \text{ (or } \leq 0 \text{)}$$

in $\mathcal{R}^n$ and generalized cone domain, respectively, and prove that the solutions must be vanish under some suitable conditions.

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1. Introduction

The study of Liouville type theorems for the solutions of linear or nonlinear hypoelliptic equations has attracted much interest in recent years [1–6].

In this paper, we aim to establish some nonlinear Liouville type theorems in general case. We first recall some concepts related to the quasi-homogeneity as follows.

Denote $x = (x_1, \cdots, x_n) \in \mathcal{R}^n$. Fix positive exponents $a_1, a_2, \cdots, a_n$ and define the dilations $\delta_\tau(x) = (\tau^{a_1} x_1, \tau^{a_2} x_2, \cdots, \tau^{a_n} x_n)$, $0 < \tau < \infty$. Let $Q = \sum_{i=1}^{n} a_i$ denote homogeneous dimension. Then by [7], there exists a norm function $r(x)$ satisfying

(1) $r(x) \in C(\mathcal{R}^n) \cap C^\infty(\mathcal{R}^n \backslash \{0\})$;

(2)$r(x) \geq 0$. Moreover, $r(x) = 0$ if and only if $x = 0$;

(3) $r(\delta_\tau(x)) = \tau r(x)$, for any $\tau > 0$;

(4) $r(x) = 1$ if and only if $\|x\| = 1$, where $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$;

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(5) Denote $a_* = \min\{a_1, \cdots, a_n\}$, $a^* = \max\{a_1, \cdots, a_n\}$. Then

$$r(x)^{a_*} \le \|x\| \le r(x)^{a^*}, \text{ for } \|x\| \ge 1;$$
$$r(x)^{a^*} \le \|x\| \le r(x)^{a_*}, \text{ for } \|x\| \le 1.$$

Suppose $P(x, \partial_x)$ is a Linear Partial Differential Operator (LPDO) on $\mathcal{R}^n$. P is said to be quasi-homogeneous of degree $m(m \in \mathcal{R})$, if for every $f \in D'(\mathcal{R}^n)$

$$P[f \circ \delta_\tau] = \tau^m P[f] \circ \delta_\tau, \text{ for all } \tau > 0.$$

Let X_j, $j = 1, \cdots, k$, be the first order smooth quasi-homogeneous vector fields on $\mathcal{R}^n$ with the property that the dimension of the Lie algebra generated by these vector fields is n at $x = 0$ (see [8]) and $X_j^* = -X_j$, $j = 1, \cdots, k$. Denote

$$L = \sum_{i=1}^{k} X_i^2.$$

Hence we have $L^t = L$. In this paper, our main theorem is the following:

Theorem 1 *Assume that there exist a nonnegative solution u of the equation*

$$Lu + f(x, u) = 0 \ (or \ \le 0) \quad in \quad \mathcal{R}^n, \tag{1}$$

where f is a nonnegative function satisfying

$$f(x, u) \ge h(x) u^p \tag{2}$$

for some function $h \ge 0$ such that, for $r(x)$ large,

$$h(x) \ge K r(x)^\gamma, \ \ K > 0, \gamma > -2. \tag{3}$$

If $1 < p \le \frac{Q+\gamma}{Q-2}$, then $u \equiv 0$.

The method of proof is rather inspired by [3–5]and [6], where the Liouville type theorems for semilinear elliptic operators and semilinear sub-Laplacian on Heisenberg group are given. In fact, the method used here can be applied to more general case and makes the proof more simple. Then, we will generalize these results to the generalized cone domains with stronger condition on h.

2. Proof of the Results

We introduce some known results which will be used later.

Lemma 1 *If P is a quasi-homogeneous LPDO of degree m with smooth coefficients on $\mathcal{R}^n$, then so is P^t with respect to the same dilations.*

This is proved by one of us in [1]. The next Lemma is evident.