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GLOBAL EXISTENCE AND EXPONENTIAL DECAY OF SOLUTIONS OF GENERALIZED KURAMOTO-SIVASHINSKY EQUATIONS*

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Abstract We study the Dirichlet initial-boundary value problem of the generalized Kuramoto-Sivashinsky equation $u_t + u_{xxxx} + \lambda u_{xx} + f(u)_x = 0$ on the interval $[0, l]$. The nonlinear function f satisfies the condition $|f'(u)| \le c|u|^{\alpha-1}$ for some $\alpha > 1$. We prove that if $\lambda < \frac{4\pi^2}{l^2}$, then the strong solution is global and exponentially decays to zero for any initial datum $u_0 \in H_0^2(0, l)$ if $1 < \alpha \le 7$, and for small $u_0 \in H_0^2(0, l)$ if $\alpha > 7$. We then consider the equation $u_t + u_{xxxx} + \lambda u_{xx} + \mu u + au_{xxx} + bu_x = F(u, u_x, u_{xx}, u_{xxx})$. We prove that if F is twice differentiable, $\nabla^2 F$ is Lipschitz continuous, and $F(0) = \nabla F(0) = 0$, and if λ and μ satisfy $\mu + \sigma(\lambda) > 0$ ($\sigma(\lambda)$=the first eigenvalue of the operator $\frac{d^4}{dx^4} + \lambda\frac{d^2}{dx^2}$), then the solution for small initial datum is global and exponentially decays to zero.

Key Words Generalized Kuramoto–Sivashinsky equations; initial-boundary value problem; global existence; exponential decay.
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1. Introduction

Kuramoto-Sivanshinsky equation, in the 1-dimensional case, reads as follows:

$$\varphi_t + \varphi_{xxxx} + \lambda\varphi_{xx} + \frac{c}{2}(\varphi_x)^2 = 0, \tag{1.1}$$

where λ and c are real constants, $\lambda > 0$, $c \ne 0$. Differentiating (1.1) in x and denoting $u = \varphi_x$, we get the following version of the 1-dimensional Kuramoto-Sivanshinsky equation:

$$u_t + u_{xxxx} + \lambda u_{xx} + cuu_x = 0. \tag{1.2}$$

The equations (1.1), (1.2) and their multi-dimensional extensions have many applications in chemistry, biology, physics etc. [1–3]. The most important parameter in these

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equations is the “anti-diffusion” coefficient λ which causes instability when its value is positive and large. Rigorious mathematical analysis of these equations has drawn great interest during the past thirty years. We refer the reader to see [4] and [5, pp. 141–150] for the mathematical treatment of the equations (1.1) and (1.2) concerning the solvability and global behavior of its initial-boundary value problems, and to see [6–8] for recent development of the research related to these equations. In particular, [8] has a quite detailed reference list for the study of these equations.

From applicational considerations we can substitute the square term $(c/2)(\varphi_x)^2$ in (1.1) with a more general nonlinear term like $f(\varphi_x)$, which gives us the equation

$$\varphi_t + \varphi_{xxxx} + \lambda\varphi_{xx} + f(\varphi_x) = 0, \tag{1.3}$$

and, by differentiating it in x and denoting $u = \varphi_x$ as before, we get

$$u_t + u_{xxxx} + \lambda u_{xx} + f(u)_x = 0. \tag{1.4}$$

Indeed, generally speaking, the experiment and real-world conditions are not so rigid that the nonlinear effect is *exactly* of the square form; other nonlinear forms are possible. Thus the equation (1.4) is a reasonable substitution of the equation (1.2). Actually, if we permit more general perturbations, then we are led to the following more open-minded generalization of the equation (1.2):

$$u_t + u_{xxxx} + \lambda u_{xx} + \mu u + a u_{xxx} + b u_x = F(u, u_x, u_{xx}, u_{xxx}). \tag{1.5}$$

Here, as in (1.2), $\lambda > 0$ is the “anti-diffusion” coefficient, μ, a and b are real constants reflecting (dispersive) linear perturbations to (1.2), and $F(u, u_x, u_{xx}, u_{xxx})$ is a nonlinear function satisfying $F(w) = o(|w|)$ as $|w| \to 0$, reflecting small nonlinear perturbations to (1.2). We call the equations (1.3)–(1.5) *generalized Kuramoto-Sivashinsky equations*.

Generalized Kuramoto-Sivashinsky equations have been studied by a number of authors; see, for instance, Biagioni *et al* [9], Guo and Jing [10], Smyrlis and Papageorgiou [11], and Zhang [12]. Related to this work we particularly mention the work of Guo and Jiang [10] and Zhang [12], where the authors established global existence and exponential decay of solutions to the Cauchy problem of certain generalized equations including (1.4). In particular, by the results of [10] and [12], we see that for the equation (1.4), if $f \in C^3(R)$ and $|f'(u)| \le c|u|^{p-1}$ for $u \in R$ and $1 < p < 7$, then for any $u \in H^2(R)$ the Cauchy problem of (1.4) has a global classical solution and $\|u(\cdot,t)\|_2$ and $\|u(\cdot,t)\|_\infty$ exponentially decay to zero as $t \to \infty$. However, for $p > 7$ the problem is open.

In this paper we shall first study the initial-boundary value problem of the equation (1.1) under the Dirichlet boundary condition. By assuming that f is continuously differentiable, f' is Lipschitz continuous, $|f'(u)| \le c|u|^{p-1}$ for some $p > 1$, and λ is less than the threshold value $4\pi^2/l^2$, we shall prove that if $1 < p \le 7$, then a global strong solution exists for any $u_0 \in H_0^2(0,l)$ and $\|u(\cdot,t)\|_{H^2(0,l)}$ converges to zero exponentially fast as $t \to \infty$, and if $p > 7$, then similar results hold for small $u_0 \in H_0^2(0,l)$. We