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NONLINEAR INSTABILITY OF EQUILIBRIUM SOLUTION FOR THE GINZBURG-LANDAU EQUATION

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Abstract We study the nonlinear instability of plane wave solutions to a Ginzburg-Landau equation with derivatives. We show that, under some condition in coefficient of the equation, these waves are unstable.

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1. Introduction

In this paper, we consider the following Ginzburg-Landau equation [1-4]

$$\begin{array}{c} W_t = \alpha_1 W_{xx} + (\lambda(|W|) + i\omega(|W|))W \\ +\alpha_3 |W|^2 W_x + \alpha_4 W^2 \overline{W}_x \end{array}, \quad x \in \Re, t > 0 \tag{1.1}$$

with the periodic initial value problem

$$\begin{cases} W(x,0) = W_0(x), \quad x \in \Re \\ W(x-D,t) = W(x+D,t), \quad D > 0, x \in \Re, t \geq 0 \end{cases} \tag{1.2}$$

where $W(x,t)$ is a complex-value function, $\alpha_j = a_j + \mathrm{i}b_j \in \Im$,

$$\begin{cases} \lambda(r) = c_1 + c_2 r^2 + c_3 r^4 \\ \omega(r) = d_1 r^2 + d_2 r^4 \end{cases} \tag{1.3}$$

with $c_j, d_j \in \Re$. For convenience, let $\alpha_1 = 1$. One of the equilibrium solutions to the Ginzburg-Landau equation is the following plane wave

$$W_p(x,t) = r_0 e^{-i\theta_0 x} \tag{1.4}$$

and

$$\begin{cases} \lambda\left(r_0\right) = \theta_0^2 - \left(b_3 - b_4\right) r_0^2 \theta_0 \\ \omega\left(r_0\right) = \left(a_3 - a_4\right) r_0^2 \theta_0 \end{cases} \tag{1.5}$$

T. Kapitula [4] shows that, as

$$\frac{2^{3/4} + \max\{1, (2/|\Gamma_3|)^{3/4}\}}{|\Gamma_3|} \left| r_0 (B_- r_0^2 - 2\theta_0) \right| < 1$$

and the initial energy $E_0 = \|W_0\|_{H^1} + \|W_0\|_{L^1}$ is small enough, these waves are nonlinear stable, where

$$B_- = b_3 - b_4 \quad and \quad \Gamma_3 = r_0 \lambda'(r_0) + 2B_- r_0^2 \theta_0 < 0$$

In present paper, we show that, under some conditions in coefficient of the equation, these waves are nonlinear unstable. We have the following main theorem.

Theorem 1.1 *Let* $\Gamma_3 > 0$ *and* $\inf\{ Re\ \lambda : \lambda \in \sigma_+(\mathcal{L})\} > 0$. *Then the plane wave solutions of the equation (1.1) is nonlinear unstable. The operator* $\mathcal{L}$ *will be defined later.*

Let

$$W\left(x,t\right) = r\left(x,t\right) e^{-i\theta(x,t)} \tag{1.6}$$

then, the equation (1.1) becomes

$$\begin{cases} r_t = r_{xx} + r\lambda\left(r\right) - r\theta_x^2 + A_+ r^2 r_x + B_- r^3 \theta_x \\ \theta_t = \theta_{xx} - \omega\left(r\right) + \dfrac{2r_x}{r}\theta_x - B_+ r r_x + A_- r^2 \theta_x, \end{cases} \tag{1.7}$$

where

$$A_\pm = a_3 \pm a_4, \quad B_\pm = b_3 \pm b_4$$

Our idea of proof is using the principle of Linearized Instability in [5] (see p.344, Theorem 9.1.3 in [5])and Theorem 9.1.3([5], p.344) under the assumption

$$\begin{cases} \sigma_+(A) = \sigma(A) \cap \{\lambda \in \mathbf{C} | Re\lambda > 0\} \neq \emptyset \\ \inf\{Re\lambda : \lambda \in \sigma_+(A)\} = w_t > 0 \end{cases} . \tag{1.8}$$

Then the problem $u'(t) = Au(t) + G(u(t)), t > 0, u(0) = u_0$ nontrivial backward solution $v \in C^\alpha([-\infty, o]; 0, w)$ with $v' \in C^\alpha([-\infty, o]; X, w)$ for every $\alpha \in [0, 1]$ and $w \in [0, w_t]$. It follows that the null solution of (1.8) is unstable, where $A : D(A) \subset X \to X$ is a linear operator such that $A : D(A) \to X$ is sectorial and the graph norm of A is equivalent to the norm of D. X is a general Banach space.

For this, we need the spectral analysis for the linearized equation.

In the paper, $\|\cdot\|_p$ represents the norm in the space $L_p(\Re)$ and $\|\cdot\|_{H^k}$ the norm in the Sobolev space $H^k(\Re)$.