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EXTREMUM PRINCIPLE FOR VERY WEAK SOLUTIONS OF $\mathcal{A}$-HARMONIC EQUATION*

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Abstract This paper deals with the very weak solutions of $\mathcal{A}$-harmonic equation

$$\mathrm{div}\mathcal{A}(x, \nabla u(x)) = 0 \tag{$*$}$$

where the operator $\mathcal{A}$ satisfies the monotonicity inequality, the controllable growth condition and the homogeneity condition. The extremum principle for very weak solutions of $\mathcal{A}$-harmonic equation is derived by using the stability result of Iwaniec-Hodge decomposition: There exists an integrable exponent

$$r_1 = r_1\left(p, n, \frac{\beta}{\alpha}\right) = \frac{1}{2}\left[p - \frac{\alpha}{100^{n^2}\beta} + \sqrt{\left(p + \frac{\alpha}{100^{n^2}\beta}\right)^2 - \frac{4\alpha}{100^{n^2}\beta}}\right]$$

such that if $u(x) \in W^{1,r}(\Omega)$ is a very weak solution of the $\mathcal{A}$-harmonic equation ($*$), and $m \le u(x) \le M$ on $\partial\Omega$ in the Sobolev sense, then $m \le u(x) \le M$ almost everywhere in Ω, provided that $r > r_1$. As a corollary, we prove that the 0-Dirichlet boundary value problem

$$\begin{cases} \mathrm{div}\mathcal{A}(x, \nabla u(x)) = 0 \\ u \in W_0^{1,r}(\Omega) \end{cases}$$

of the $\mathcal{A}$-harmonic equation has only zero solution if $r > r_1$.

Key Words $\mathcal{A}$-harmonic equation; extremum principle; very weak solution; Iwaniec-Hodge decomposition.

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1. Introduction

Throughout this paper Ω will stands for a bounded regular domain in $\mathrm{R}^n, n \geq 2$. By regular domain we understand any domain of finite measure for which the estimates for the Iwaniec-Hodge decomposition in Lemma 1 and Lemma 2 are justified. See [1] and [2]. A Lipschitz domain, for example, is regular. We shall examine the following divergence type elliptic equation (also called $\mathcal{A}$-harmonic equation)

$$\mathrm{div}\mathcal{A}(x, \bigtriangledown u(x)) = 0 \tag{1}$$

where $\mathcal{A} : \Omega \times \mathrm{R}^n \to \mathrm{R}^n$ satisfies the usual measurability conditions (Carathéodory conditions) and that for some $1 < p < \infty$, the following conditions hold:
(i) the monotonicity inequality

$$\langle \mathcal{A}(x, \xi), \xi \rangle \geq \alpha |\xi|^p$$

(ii) the controllable growth condition

$$|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}$$

(iii) the homogeneity condition

$$\mathcal{A}(x, \lambda\xi) = |\lambda|^{p-2}\lambda\mathcal{A}(x, \xi)$$

for almost every $x \in \Omega$ and all $\xi \in \mathrm{R}^n$, $0 < \alpha \leq \beta < \infty$, $\lambda \in \mathrm{R}$.

Remark The mapping $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$, which generates the p-harmonic equation

$$\mathrm{div}| \bigtriangledown u(x)|^{p-2} \bigtriangledown u(x) = 0$$

satisfies the assumptions (i), (ii) and (iii).

Definition 1 *A weak solution of (1) is defined as* $u \in W^{1,p}(\Omega)$ *satisfy*

$$\int_\Omega \langle \mathcal{A}(x, \bigtriangledown u(x)), \bigtriangledown \phi(x) \rangle dx = 0 \tag{2}$$

for every $\phi \in C_0^\infty(\Omega)$.

The p-integrability of $\bigtriangledown u(x)$ is not required for (2) to be of sense, but it is a *natural* assumption because it is used in studying regularity of weak solutions. Actually, the properties of weak solutions are often deduced by a suitable choice of test function in (2), typically $\phi(x) = \lambda(x)u(x)$, with $\lambda(x)$ a cut-off function. For example, the well-known higher integrability of $\bigtriangledown u(x)$ proved first in [3] is achieved by the technique of reverse Hölder inequalities which are obtained by testing (2) with appropriate $\phi(x)$. See also [4] and references therein.

In the paper [2], the notion of *very weak* solution is considered, relaxing the natural integrability assumption.