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L^p-REGULARITY FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

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Abstract We study here a class of pseudodifferential operators with weighted symbols of Shubin type. First, we develop the basic elements of the pseudodifferential calculus for these operators, proving in particular a result of L^p-boundedness. Then we derive regularity results in the frame of suitably defined functional spaces of Sobolev type.

Key Words Pseudo-differential operators; L^p boundedness; global hypoellipticity.

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1. Introduction

In this paper we shall consider a class of pseudodifferential operators, including the parametrices of the linear partial differential operators of Schrödinger type $P(x, D) = -\Delta + V(x)$ where the potential $V(x) = \sum_{\alpha\in\mathcal{R}} a_\alpha x^\alpha$ is a multi-quasi-elliptic polynomial with constant coefficients $a_\alpha \in \mathbb{C}$, with $\mathcal{R}$ complete polyhedron of $\mathbb{R}^n$; cf. [1-8].

We will summarize later the notion of complete polyhedron of $\mathbb{R}^n$ and multi-quasi-ellipticity. However, to be more clear, we give here two explicit examples of such $V(x)$ in two variables (x_1, x_2):

$$V_1(x_1, x_2) = x_1^h + x_2^k, \tag{1}$$

with k, h even positive integers;

$$V_2(x_1, x_2) = x_1^{h_1} + x_1^{k_1}x_2^{k_2} + x_2^{h_2}, \tag{2}$$

with h_i, k_i even positive integers, $h_i > k_i > 0$ $(i = 1, 2)$ and $\frac{k_1}{h_1} + \frac{k_2}{h_2} > 1$.

By arguing in the framework of the pseudodifferential operators with symbol in the weighted Shubin classes (see [9-12]) it can be shown that $u \in L^2(\mathbb{R}^n)$, $\partial^2_{x_j} u \in L^2(\mathbb{R}^n)$ and $x^\alpha u \in L^2(\mathbb{R}^n)$ for any $j = 1, ..., n$ and $\alpha \in \mathcal{R}$, provided that $P(x, D)u \in L^2(\mathbb{R}^n)$, cf. the preceding examples.

In this paper, the above L^2-regularity result is extended to any $1 < p < \infty$ in the context of a calculus of L^p-bounded weighted pseudodifferential operators.

As is well-known, pseudodifferential operators with symbol in the Shubin class $\Gamma^0_{\rho,\Lambda}$, cf. [9, 10], with weight function $\Lambda(x,\xi)$ and $\rho = \rho(\Lambda) \in (0,1]$, are not in general L^p-bounded for $p \neq 2$ (see [13] for instance).

Here we deal with suitable subclasses $M\Gamma^m_{\rho,\Lambda}$ of the Shubin classes which are obtained by assuming an additional Lizorkin-Marcinkiewicz type condition on the decay of the derivatives of the symbol. By means of the Lizorkin-Marcinkiewicz theorem about Fourier multipliers ([14]) the related pseudodifferential operators are L^p-continuous when the order m is zero; cf. [15, 16], for the related results in a local context.

In Sections 2 and 3, the class of the weight functions and the related weighted symbol classes are introduced. The basic elements of the symbolic calculus in this framework are developed and some properties of the weighted-elliptic symbols are also considered.

In Sections 4 and 5 various kind of pseudodifferential operators with symbol in $M\Gamma^m_{\rho,\Lambda}$ and their algebra properties are investigated, while Section 6 is devoted to the construction of the parametrix of an elliptic operator; since here we follow closely the lines of the standard symbolic calculus, cf. [9], the proofs are usually omitted or outlined only.

The main L^p-continuity result for operators of order zero is proved in detail in Section 7 and in Section 8 the action of operators with arbitrary order on a scale of suitably defined weighted Sobolev spaces is derived.

In Section 9 we consider the same operators acting on the functional spaces defined as the images, under the inverse Fourier transform, of the Sobolev spaces in Section 8. The symmetric behavior of symbols in $M\Gamma^m_{\rho,\Lambda}$ with respect to the variables x, ξ leads to the continuity of the related pseudodifferential operators on such spaces.

In Section 10, the regularity results for weighted-elliptic pseudodifferential operators are stated in the frame of all the previous functional spaces; as an example they are applied in particular to the Schrödinger type operators mentioned at the beginning.

We finally observe that, as far as we know, our results are new also in the homogeneous classes of [11, 12], corresponding to the weight $\Lambda(x,\xi) = \sqrt{|x|^2 + |\xi|^2 + 1}$ and having as basic example the harmonic oscillator $-\Delta + |x|^2$; however in this case L^p-regularity of the solutions can be deduced easily from known results, cf. [17].

2. Weight Functions

Definition 2.1 *We say that a positive function $\Lambda(z) \in C^\infty(\mathbb{R}^N)$ is a weight function if the following conditions are fulfilled:*

1. *there exist positive constants $\mu_0 \leq \mu_1$ and C such that*

$$C^{-1}(1+|z|)^{\mu_0} \leq \Lambda(z) \leq C(1+|z|)^{\mu_1}, \tag{3}$$

for all $z \in \mathbb{R}^N$;

2. *for every $t \in \mathbb{R}^N$ there exists $C = C_t > 0$ such that*

$$\Lambda(tz) \leq C\Lambda(z), \tag{4}$$