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A CRITICAL VALUE FOR GLOBAL NONEXISTENCE OF SOLUTION OF A WAVE EQUATION*

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Abstract Consider the Cauchy problem for a wave equation on R^2: $u_{tt} - \Delta u = |u|^{p-1}u$. In 1981 Glassey gave a guess to a critical value $p(2) = \frac{1}{2}\left(3+\sqrt{17}\right)$: when $p > p(2)$ there may exist a global solution and when $1 < p < p(2)$ the solution may blow up. By our main result in this paper a counter example to the guess is given that the solution may also blow up in finite time even if $p(2) < p < 5$.
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Consider the Cauchy problem for a wave equation on R^2:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - \Delta u = |u|^{p-1}u, & x \in R^2, \quad 0 < t < T, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in R^2, \end{cases} \tag{1}$$

where we assume that

(H1) $f(x),\, g(x) \in C_0^\infty(R^2)$, $\text{supp}\{f, g\} \subset \{\|x\| \le L\}$,

$$\int_{R^2} f\mathrm{d}x > 0, \qquad \int_{R^2} g\mathrm{d}x > 0.$$

Theorem(Glassey[1]) *When* $1 < p < p(2) = \frac{1}{2}\left(3+\sqrt{17}\right)$, $T < +\infty$, *i.e the solution of (1) may blow up in finite time* $T < +\infty$.

In Case R^3, John[2] gave the critical value $p(3) = 1 + \sqrt{2}$: when $1 < p < p(3)$ the solution may blow up and when $p > p(3)$ there may exist global solution. Thus, Glassey gave a guess that $p(2)$ may also be a critical value for blow up: there may exist a global solution if $p > p(2)$. But a counter example can be given to Glassey's guess by our following main result.

Theorem *Let* $u(x,t) \in C^2(R^2 \times [0,T])$ *be a nontrivial solution of (1) with finite speed of propagation. Assume that*

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(H1) *The same as Glassey Theorem, and* $\int_{R^2} fg\mathrm{d}x > 0,\ f(x) \not\equiv 0,$

(H2) $3 < p < 5,$

(H3) $I_0 = \frac{2}{p+1}\int_{R^2} |f|^{p+1}\mathrm{d}x - \left[\int_{R^2}\left(|\nabla f|^2 + |g|^2\right)\mathrm{d}x\right] \geq 0.$

Then $T < +\infty$, *i.e the solution of (1) may blow up in finite time* $T < +\infty$.

Remark It is well known that (H1) implies the existence of a unique classical solution to (1).

Proof We will estimate $F(t) = \int_{R^2} u^2(x,t)\mathrm{d}x$ by using the method similar to[3].

First, multiplying the equation (1) by $u(x,t)$ and integrating over R^2, we have

$$\frac{1}{2}F''(t) = \frac{p-1}{p+1}\int_{R^2}|u|^{p+1}\mathrm{d}x + \frac{2}{p+1}\int_{R^2}|u|^{p+1}\mathrm{d}x + \int_{R^2}|u_t|^2\mathrm{d}x - \int_{R^2}|\nabla u|^2\mathrm{d}x. \quad (2)$$

Next, multiplying the equation (1) by u_t and integrating over $R^2 \times [0,t]$ we have

$$\int_{R^2}|u_t|^2\mathrm{d}x = \frac{2}{p+1}\int_{R^2}|u_t|^{p+1}\mathrm{d}x - \int_{R^2}|\nabla u|^2\mathrm{d}x - I_0. \quad (3)$$

By (H3), (2) and (3) yield

$$\frac{1}{2}F''(t) = \frac{p-1}{p+1}\int_{R^2}|u|^{p+1}\mathrm{d}x + 2\int_{R^2}|u_t|^2\mathrm{d}x + I_0. \quad (4)$$

Thus $F''(t) \geq 0$ and $F'(t)$ is monotone nondecreasing. Therefore $F'(t) \geq F'(0) > 0$ by (H1), and $F(t)$ is also monotone nondecreasing , thus $F(t) \geq F(0) = \int_{R^2} f^2\mathrm{d}x > 0$.

Now, by finite speed of propagation and by (H1), we have

$$F(t) = \int_{R^2} u^2\mathrm{d}x = \int_{\|x\|\leq t+L} u^2\mathrm{d}x \leq \left\{\int_{\|x\|\leq t+L}|u|^{p+1}\mathrm{d}x\right\}^{\frac{2}{p+1}}\left\{\int_{\|x\|\leq t+L} 1\mathrm{d}x\right\}^{\frac{p-1}{p+1}}$$

i.e.

$$F(t)^{\frac{p+1}{2}} \leq \pi^{\frac{p-1}{2}}(t+L)^{p-1}\int_{R^2}|u|^{p+1}\mathrm{d}x. \quad (5)$$

Combining (5) with (4), we obtain

$$F''(t) \geq C_0(t+L)^{1-p}F(t)^{\frac{p+1}{2}} \quad (6)$$

but $F(t) \geq F(0) = \int_{R^2} f^2\mathrm{d}x > 0$, thus

$$F''(t) \geq k_0(t+L)^{1-p},$$

so

$$F'(t) \geq F'(0) + \frac{k_0}{2-p}(t+L)^{2-p} - \frac{k_0}{2-p}L^{2-p}.$$