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EXISTENCE AND NONUNIQUENESS OF WEAK SOLUTIONS FOR A DEGENERATE DIFFUSION EQUATION*

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Abstract In this paper, we prove the existence and nonuniqueness of the weak solutions of the initial and boundary value problem for a nonlinear degenerate parabolic equation not in divergence form. Localization property of weak solutions will be also discussed.
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1. Introduction

In this paper, we are concerned with the initial and boundary value problem

$$\begin{cases} u_t = u\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \gamma|\nabla u|^p & \text{in } \Omega_T, & (1.1)\\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T), & (1.2)\\ u(x,0) = u_0(x) & \text{in } \Omega, & (1.3)\end{cases}$$

where $\Omega_T = \Omega\times(0,T)$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, $p\geq 2,\gamma$ are constants and u_0 satisfies

$$(\mathrm{H})_1 \qquad 0\leq u_0\in C(\overline{\Omega}),\quad u_0 = 0 \text{ on } \partial\Omega.$$

(1.1) is an equation not in divergence form, which degenerates whenever $u = 0$ or $\nabla u = 0$. In general, the problem does not admit classical solutions. Therefore we need to consider its weak solutions. Moreover, we are interested in nonnegative solutions.

Notice that (1.1) is closely related to the filtration equation (see for example [1-3])

$$v_t = \mathrm{div}\{|\nabla v^m|^{p-2}\nabla v^m\} \quad (m\neq 0). \tag{1.4}$$

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Indeed, if $m(p-1)-1\neq 0$ and let

$$\begin{cases} \gamma = \dfrac{p-1}{m(p-1)-1}, \\ u = m|\dfrac{p-1}{m(p-1)-1}|^{(p-2)/(p-1)} v^{(m(p-1)-1)/(p-1)}, \end{cases}$$

then (1.1) can be transformed formally into (1.4). The corresponding relations are

$$\begin{cases} \gamma > 0 \text{ corresponds to } m > \dfrac{1}{p-1} \text{ (slow diffusion);} \\ \gamma < 0 \text{ corresponds to } m < \dfrac{1}{p-1} \text{ (fast diffusion);} \\ \gamma = 0 \text{ corresponds to the limit case when } m \to \infty. \end{cases}$$

There are some papers devoted to (1.1) with $p = 2$, namely, the equation

$$u_t = u\Delta u + \gamma|\nabla u|^2. \tag{1.5}$$

In the case $\gamma = 0$ Dal Passo and Luckhaus[4] discussed the existence, nonuniqueness and localization property of the weak solutions of the initial and boundary value problem for (1.5), and the nonuniqueness phenomenon of the weak solutions is also discovered independently by Ughi[5]. Some other results for (1.5) can be referred to [6-12]. However, there have been few papers dealing with (1.1) with $p \neq 2$. In this paper, we study the case

(H)$_2$ $$\gamma \in (1-p, 0].$$

Some results are obtained under the additional condition

(H)$_3$ $$|\nabla u_0^{1+\gamma/(p-1)}| \in L^p(\Omega_T).$$

Definition 1.1 *A nonnegative function u is called a weak solution of the problem (1.1)-(1.3) if*

(*a*) $u \in L^\infty(\Omega_T) \cap L^p(0,T;W_0^{1,p}(\Omega))$,

(*b*) $\displaystyle\iint_{\Omega_T} (-u\varphi_t + u|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + (1-\gamma)|\nabla u|^p\varphi)dxdt = 0, \forall\varphi \in C_0^\infty(\Omega_T)$,

(*c*) $\displaystyle\lim_{t\to 0^+}\int_\Omega |u(x,t)-u_0(x)|dx = 0$.

Remark 1.1 If the boundary value function is an arbitrary nonnegative function $\psi \in L^\infty(\Omega_T)\cap L^p(0,T;W^{1,p}(\Omega))$, then we define the weak solutions of the initial and boundary value problem by requiring $u-\psi \in L^\infty(\Omega_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$ instead of $u \in L^\infty(\Omega_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$.

Remark 1.2 For any weak solution u with $u_t \in L^2(\Omega_T)$, the integral identity (b) in Definition 1.1 can be rewritten as

$$\iint_{\Omega_T}(u_t\varphi + u|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + (1-\gamma)|\nabla u|^p\varphi)dxdt = 0, \forall\varphi\in C_0^\infty(\Omega_T).$$