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# EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR A NONLINEAR PARABOLIC EQUATION RELATED TO IMAGE ANALYSIS*

Wang Lihe
( Department of Mathematics, University of Iowa, Iowa City, USA)
(E-mail: lwang@math.uiowa.edu)
Zhou Shulin
(LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China)
(E-mail: szhou@math.pku.edu.cn)

Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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**Abstract** In this paper we establish the existence and uniqueness of weak solutions for the initial-boundary value problem of a nonlinear parabolic partial differential equation, which is related to the Malik-Perona model in image analysis.

**Key Words** Existence; uniqueness; nonlinear parabolic partial differential equations.
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## 1. Introduction

Suppose that $\Omega$ is a bounded, open domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $T$ is a positive number. In this paper we study the following nonlinear parabolic initial-boundary value problem

$$\begin{cases} u_t - \operatorname{div}\Big(\Phi'(|\nabla u|)\dfrac{\nabla u}{|\nabla u|}\Big) = 0 & \text{in} \quad \Omega \times (0,T], \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\Phi(s) = s\log(1+s)$ $(s \geq 0)$, and $\mathbf{n}$ is the unit normal vector of $\partial\Omega$.

In image analysis, denoising and segmentation are two intertwined important goals. The heat equation is usually used to deniose images since it is the limit form of the

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holomorphic filters (see [1]). The reason is that there is a strong regularizing effect in the heat equation so that the noise would be removed from an image. However, on the other hand, the regularizing effect in the heat equation is so strong that solutions of the heat equation will be infinitely smooth after arbitrarily small positive time. Thus the edges of the image will be greatly blurred when the heat equation is used to denoise the image. Therefore, the heat equation is not a good option to denoise and segment images. In 1990, Perona and Malik invented the Malik-Perona model

$$\begin{cases} u_t - \operatorname{div}\big(c(|\nabla u|^2)\nabla u\big) = 0 & \text{in} \quad \Omega \times (0,T], \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where $\Omega$ is an image domain in $\mathbb{R}^2$ and $c(s) > 0$, to replace the heat equation. Now this model is well-known and has been widely used to denoise and segment images. In order to understand their idea, we introduce a tangent unit vector $\mathbf{T} = \nabla u/|\nabla u|$ and a unit normal vector $\mathbf{N}$ perpendicular to $\mathbf{T}$. Then (1.2) reads as

$$u_t = c(|\nabla u|^2)u_{TT} + b(|\nabla u|^2)u_{NN}, \tag{1.3}$$

with $b(s) = c(s) + 2sc'(s)$. Therefore, equation (1.3) may be interpreted as a sum of a diffusion in the $\mathbf{T}$-direction plus a diffusion in the $\mathbf{N}$-direction. On the region where the gradient $\nabla u$ is small, it is expected that equation (1.3) behaves like the heat equation. Generally speaking, in the neighborhood of an edge E, the image presents a large gradient. Therefore, on the region where the gradient $\nabla u$ is large, it is expected that the diffusion is anisotropic to preserve the edge. Thus, it is preferable to diffuse much stronger along E (in the tangent direction) than across the edge (in the normal direction). That means, it would be desirable to smooth the function $u$ much more in the tangential direction $\mathbf{T}$ than in the normal direction $\mathbf{N}$ as the normal direction $\mathbf{N}$ is usually perpendicular to the edges. Thus the condition $\lim\limits_{s\to+\infty} b(s)/c(s) = 0$ is usually assumed, or equivalently

$$\lim_{s\to+\infty} \frac{sc'(s)}{c(s)} = -\frac{1}{2}. \tag{1.4}$$

In the meanwhile, the condition $c(0) = 1$ is assumed without loss of generality. Summarizing the above reasoning, the assumptions imposed on $c(s)$ in equation (1.2) are usually

$$\begin{cases} c : [0,+\infty) \to (0,+\infty) \quad \text{decreasing}, \\ c(0) = 1, \quad c(s) \approx \dfrac{1}{\sqrt{s}} \quad \text{as} \quad s \to +\infty, \\ b(s) = c(s) + 2sc'(s) > 0. \end{cases} \tag{1.5}$$

A canonical example of a function $c(s)$ satisfying (1.5) is $c(s) = 1/\sqrt{1+s}$. The corre-