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MAXIMUM PRINCIPLES OF NONHOMOGENEOUS SUBELLIPTIC P-LAPLACE EQUATIONS AND APPLICATIONS*

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Abstract Maximum principles for weak solutions of nonhomogeneous subelliptic p-Laplace equations related to smooth vector fields $\{X_j\}$ satisfying the Hörmander condition are proved by the choice of suitable test functions and the adaption of the classical Moser iteration method. Some applications are given in this paper.
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1. Introduction

Over the last decades, the study of nonelliptic equations arising from general families of non-commuting vector fields has made a great development. In spite of the formidable progress, there is still much to discover concerning the basic properties of solutions to these classes of equations.

Consider a family of C^∞ vector fields $X_1, \cdots, X_N$ in $\mathbb{R}^n$, and assume that Hörmander finite rank condition [1]

$$\text{rank Lie } [X_1, \cdots, X_N] = n \tag{1.1}$$

is satisfied at each $x \in \mathbb{R}^n$. In this paper we are concerned with a kind of the so-called subelliptic p-Laplace equation:

$$\sum_{j=1}^{N} X_j^* \left(|Xu|^{p-2} X_j u \right) = 0, \tag{1.2}$$

where X_j^* denotes the formal adjoint of X_j, $Xu = (X_1 u, \cdots, X_N u)$ is the subelliptic gradient of u and $1 \le p < \infty$ is fixed. It appears in the study of quasiregular mappings

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in stratified Lie groups, also known as Carnot groups [2]. We note that (1.2) is the Euler-Lagrange equation of the Sobolev functional

$$J_p(u) = \int |Xu|^p \, \mathrm{d}x. \tag{1.3}$$

When $p = 2$, (1.2) is the Hömander type equation

$$\sum_{j=1}^{N} X_j^* X_j u = 0. \tag{1.4}$$

An important result in the study of (1.4) was given in Nagel, Stein and Wainger's famous paper [3], in which the following estimates for the Carnot-Carathéodory metric balls were proved: for every $\mathbf{K} \subset\subset \mathbb{R}^n$, there exist positive constants C, R_0 and Q such that, for any $x \subset \mathbf{K}$, $0 < r < R_0$, and $0 < t < 1$,

$$|B_d(x, tr)| \geq Ct^Q \, |B_d(x, r)| \,, \tag{1.5}$$

where $B_d(x, r) = \{y \in \mathbb{R}^n \big| d(x, y) < r\}$ is the ball relative to the control distance d associated to the vector fields $X_1, \cdots, X_N$. The number Q plays the role of a dimension in the local analysis of (1.4). It will be called the homogeneous dimension of $\mathbf{K}$ with respect to the family $X_1, \cdots, X_N$.

In [2], a strong maximum principle of homogeneous subelliptic equations is given with the Hölder estimate. Gutiérrez and Lanconelli in [4] proved a maximum principle and Harnack inequalities for second order uniformly X-elliptic operators. Xu has studied some subelliptic equations associated with the vector fields satisfying Hörmander condition. He obtained regularity for quasilinear subelliptic equations in [5] and Sobolev inequality of these vector fields in [6]. Primarily inspired by [4], our purpose is to establish a maximum principle for the nonhomogeneous equation

$$L_p u = -\sum_{j=1}^{N} X_j^* (|Xu|^{p-2} X_j u) = f(x), \tag{1.6}$$

on the bounded open subset in $\mathbb{R}^n$. Although the method we used is similar to that of [4], the question we discussed here is nonlinear in substance.

We introduce some definitions and results that will be needed in the sequel. Solutions to (1.6) shall be understood in a suitable weak sense. Throughout the paper, Ω denotes a bounded open subset in $\mathbb{R}^n$ and Q is the homogeneous dimension of Ω relative to $X_1, \cdots, X_N$.

Let $S^{1,p}(\Omega)$ be the closure of $\{u \in C^\infty(\Omega) : u, X_j u \in L^p(\Omega), \text{ for } 1 \leq j \leq N\}$ under the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left[\int_\Omega (|u|^p + |Xu|^p) \mathrm{d}x \right]^{\frac{1}{p}}. \tag{1.7}$$