THE LIMITING ABSORPTION METHOD FOR A TRANSMISSION PROBLEM IN ACOUSTIC SCATTERING

Messaoud SOUILAH
( Faculté de Mathématiques, USTHB
Bp 32 El Alia Bab Ezzouar, Alger 16111, Algérie)
(E-mail: souilahdz@yahoo.fr)
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Abstract  The limiting absorption principle is used to solve the scattering problem of time harmonic acoustic waves by penetrable objects in Sobolev spaces. The method is based on integral representation of the solution using the Green’s kernel of the Helmholtz equation.

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1. Position of the Problem

Let $\Omega_1$ be an open bounded domain in $\mathbb{R}^n$ constituting an obstacle of boundary $\Gamma = \partial \Omega_1 \in C^\infty$ placed in an infinite medium, let $\Omega_2 = \mathbb{R}^n \setminus \overline{\Omega_1}$ its outside. One notes by $\rho_1, \rho_2$ the respective densities of the mediums $\Omega_1, \Omega_2$ and by $\chi$ a constant function per pieces such that $\chi = 1/\rho_1$ in $\Omega_1$ and $\chi = 1/\rho_2$ in $\Omega_2$.

One sends in direction of the obstacle $\Omega_1$ a plane and harmonic incidental wave of the form $U(x, t) = e^{-i\omega t}u_0(x)$. The wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta_x U = 0 \text{ in } \Omega_2$$

(1.1)

gives the following Helmholtz equation, called also the reduced wave equation

$$\Delta u_0 + k_2^2 u_0 = 0,$$

(1.2)

where $k_1 = \omega/c_1, k_2 = \omega/c_2$ are the respective wave numbers in $\Omega_1, \Omega_2$, $\omega$ the pulsation and $c_1, c_2$ the speeds of sound in $\Omega_1, \Omega_2$.

After its interaction with the obstacle, the incidental wave $u_0$ is partially transmitted in $\Omega_1$ and partially reflected. One notes by $v$ the diffracted wave

$$v = \begin{cases} v_1 \text{ in } \Omega_1 \\ v_2 \text{ in } \Omega_2 \end{cases}$$

(1.3)
and by \( u = u_0 + v \) the total wave. The determination of the unknown \( v \) remains to the resolution of the following problem ([1-4])

\[
\begin{cases}
\text{Find } v \in C^\infty(\Omega_1) \cap C^\infty(\Omega_2) \\
\Delta v + k_1^2 v = (k_2^2 - k_1^2) u_0 \text{ in } \Omega_1 \\
\Delta v + k_2^2 v = 0 \text{ in } \Omega_2 \\
[v] = 0 \text{ on } \Gamma \\
[\chi \frac{\partial v}{\partial n}] = -[\chi \frac{\partial u_0}{\partial n}] \text{ on } \Gamma \\
\frac{\partial v}{\partial r} - ik_2 v = o \left( r^{\frac{1-n}{2}} \right) , \ r \to +\infty
\end{cases}
\]

where the condition at infinity \( \frac{\partial v}{\partial r} - ik_2 v = o \left( r^{\frac{1-n}{2}} \right) \text{ as } r = |x| \to +\infty \), is supposed to be uniform with respect to the angular variables \( x/|x| \), and is known as the Sommerfeld radiation condition. It represents the decrease of the energy at infinity.

\section{2. The Transmission Problem}

In order to give a functional framework to the previous problem and taking into account the fact that the solutions of the Helmholtz equation are not in \( L^2(\Omega') \) if \( \Omega' \) is an exterior domain, one states the following more general problem (\( k_1, k_2, \rho_1, \rho_2 \) being strictly positive constants).

Let \( f \in L^2(\Omega_1) \), \( g \in H^{1/2}(\Gamma) \) and \( \zeta \in H^{-1/2}(\Gamma) \), one seeks \( v \) such as

\[
\begin{cases}
\text{Find } v \in H^1(\Omega_1) \cap H^1_{\text{loc}}(\Omega_2) \\
\Delta v + k_1^2 v = -f \text{ in } \Omega_1 \\
\Delta v + k_2^2 v = 0 \text{ in } \Omega_2 \\
[v] = g \text{ on } \Gamma \\
[\chi \frac{\partial v}{\partial n}] = \zeta \text{ on } \Gamma \\
\frac{\partial v}{\partial r} - ik_2 v = o \left( r^{\frac{1-n}{2}} \right) , \ r \to +\infty
\end{cases}
\]

\textbf{Remark} One can suppose \( g \equiv 0 \) on \( \Gamma \) by introducing the problem

\[
\begin{cases}
w \in H^1(\mathbb{R}^n) \\
\Delta w = 0 \text{ in } \Omega_1 \\
w = g \text{ on } \Gamma
\end{cases}
\]

while replacing \( v \) by \( v - \bar{w} \), where \( \bar{w} \) is the prolongation of \( w \) by 0 to \( \mathbb{R}^n \). The previous problem becomes