J. Partial Diff. Eqs. **19**(2006), 359–368
©International Academic Publishers Vol.19 No.4

THE LIMITING ABSORPTION METHOD FOR A TRANSMISSION PROBLEM IN ACOUSTIC SCATTERING

Messaoud SOUILAH
(Faculté de Mathématiques, USTHB
Bp 32 El Alia Bab Ezzouar, Alger 16111, Algérie)
(E-mail: souilahdz@yahoo.fr)
(Received July. 31, 2005; revised Mar. 19, 2006)

Abstract The limiting absorption principle is used to solve the scattering problem of time harmonic acoustic waves by penetrable objects in Sobolev spaces. The method is based on integral representation of the solution using the Green's kernel of the Helmholtz equation.
Key Words Limiting absorption principle; Helmholtz equation; exterior domains; Sommerfeld radiation condition; Rellich lemma.
2000 MR Subject Classification 65R20, 78A45, 76M15.
Chinese Library Classification O175.5.

1. Position of the Problem

Let Ω_1 be an open bounded domain in $\mathbb{R}^n$ constituting an obstacle of boundary $\Gamma = \partial\Omega_1 \in C^\infty$ placed in an infinite medium, let $\Omega_2 = \mathbb{R}^n \setminus \overline{\Omega_1}$ its outside. One notes by ρ_1, ρ_2 the respective densities of the mediums Ω_1, Ω_2 and by χ a constant function per pieces such that $\chi = 1/\rho_1$ in Ω_1 and $\chi = 1/\rho_2$ in Ω_2.

One sends in direction of the obstacle Ω_1 a plane and harmonic incidental wave of the form $U(x,t) = e^{-i\omega t}u_0(x)$. The wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta_x U = 0 \text{ in } \Omega_2 \tag{1.1}$$

gives the following Helmholtz equation, called also the reduced wave equation

$$\Delta u_0 + k_2^2 u_0 = 0, \tag{1.2}$$

where $k_1 = \omega/c_1,\ k_2 = \omega/c_2$ are the respective wave numbers in $\Omega_1, \Omega_2,\ \omega$ the pulsation and c_1, c_2 the speeds of sound in Ω_1, Ω_2.

After its interaction with the obstacle, the incidental wave u_0 is partially transmitted in Ω_1 and partially reflected. One notes by v the diffracted wave

$$v = \begin{cases} v_1 \text{ in } \Omega_1 \\ v_2 \text{ in } \Omega_2 \end{cases} \tag{1.3}$$

and by $u = u_0 + v$ the total wave. The determination of the unknown v remains to the resolution of the following problem ([1-4])

$$\begin{cases} \text{Find } v \in C^\infty(\overline{\Omega_1}) \cap C^\infty(\overline{\Omega_2}) \\ \Delta v + k_1^2 v = (k_2^2 - k_1^2)u_0 \text{ in } \Omega_1 \\ \Delta v + k_2^2 v = 0 \text{ in } \Omega_2 \\ [v] = 0 \text{ on } \Gamma \\ [\chi \frac{\partial v}{\partial n}] = -[\chi \frac{\partial u_0}{\partial n}] \text{ on } \Gamma \\ \frac{\partial v}{\partial r} - ik_2 v = o\left(r^{\frac{1-n}{2}}\right), \; r \to +\infty \end{cases} \tag{1.4}$$

where the condition at infinity $\frac{\partial v}{\partial r} - ik_2 v = o\left(r^{\frac{1-n}{2}}\right)$ as $r = |x| \to +\infty$, is supposed to be uniform with respect to the angular variables $x/|x|$, and is known as the Sommerfeld radiation condition. It represents the decrease of the energy at infinity.

2. The Transmission Problem

In order to give a functional framework to the previous problem and taking into account the fact that the solutions of the Helmholtz equation are not in $L^2(\Omega')$ if Ω' is an exterior domain, one states the following more general problem (k_1, k_2, ρ_1, ρ_2 being strictly positive constants).

Let $f \in L^2(\Omega_1),\; g \in H^{1/2}(\Gamma)$ and $\zeta \in H^{-1/2}(\Gamma)$, one seeks v such as

$$(P) \quad \begin{cases} v \in H^1(\Omega_1) \cap H^1_{loc}(\overline{\Omega_2}) \\ \Delta v + k_1^2 v = -f \text{ in } \Omega_1 \\ \Delta v + k_2^2 v = 0 \text{ in } \Omega_2 \\ [v] = g \text{ on } \Gamma \\ [\chi \frac{\partial v}{\partial n}] = \zeta \text{ on } \Gamma \\ \frac{\partial v}{\partial r} - ik_2 v = o\left(r^{\frac{1-n}{2}}\right), \; r \to +\infty \end{cases} .$$

Remark One can suppose $g \equiv 0$ on Γ by introducing the problem

$$\begin{cases} w \in H^1(\mathbb{R}^n) \\ \Delta w = 0 \text{ in } \Omega_1 \\ w = g \text{ on } \Gamma \end{cases} , \tag{2.1}$$

while replacing v by $v - \widetilde{w}$, where $\widetilde{w}$ is the prolongation of w by 0 to $\mathbb{R}^n$. The previous problem becomes