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CONCERNING TIME-PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN CYLINDRICAL DOMAINS UNDER NAVIER BOUNDARY CONDITIONS

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Abstract The problem of the existence of time-periodic flows in infinite cylindrical pipes in correspondence to any given, time-periodic, total flux, was solved only quite recently in [1]. In this last reference we solved the above problem for flows under the non-slip boundary condition as a corollary of a more general result. Here we want to show that the abstract theorem proved in [1] applies as well to the solutions of the well known slip (or Navier) boundary condition (1.7) or to the mixed boundary condition (1.14). Actually, the argument applies for solutions of many other boundary value problems. This paper is a continuation of reference [1], to which the reader is referred for some notation and results.
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1. Introduction

Let Ω be a bounded, regular, connected open set in $\mathbb{R}^n$, $n \geq 1$, and consider a cylindrical, $(n+1)-$dimensional infinite pipe $\Lambda = \Omega \times \mathbb{R}$. We denote by Γ the boundary of Ω and by $\Sigma = \Gamma \times \mathbb{R}$ the boundary of Λ. We set $x = (x_1, ..., x_n)$ and denote by z the longitudinal coordinate along the axis of the pipe, say $z = x_{n+1}$. We denote by χ the component of the velocity v in the axial direction z. Note that the physical dimension is $N = n+1$. In the sequel we consider the problem of the existence of T-periodic flows of the Navier-Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\,\Delta\,v + (v\cdot\nabla)\,v + \nabla p = 0, \\ \nabla \cdot v = 0 \quad \text{in } \Lambda \times \mathbb{R}_t\,, \end{cases} \tag{1.1}$$

under the slip boundary condition (1.7) or the mixed boundary condition (1.8), in the cylindrical domain Λ. Here the differential operators Δ and ∇ act on all the variables

$(x_1, ..., x_n,\, z)$ and there is a (arbitrarily given) T-periodic total flux $g(t)$, i.e.,

$$\int_\Omega \chi(x,z,t)\,dx = g(t) \tag{1.2}$$

for each cross section $\Omega(z) = \{(x,z) : x \in \Omega\}$. We call the flux $g(t)$, in the cross sections of the pipe, the *total flux.*

In reference [1] we considered the non-slip boundary condition

$$v = 0 \quad \text{on } \Sigma \times \mathbb{R}_t \tag{1.3}$$

and proved that to each T-periodic total flux $g(t)$ there corresponds one and only one T-periodic flow, parallel to the z-axis, independent of z and satisfying the flux constraint (1.2). See Theorem 1 in reference [1]. Particularly related papers are [2] by Galdi and Robertson and [3] by Pileckas. In reference [2], drawing on [1], the authors show an interesting relation between the total flux and the pressure gradient for the solutions to the above problem. In reference [3] the author considers the initial-boundary value problem for the solutions of (1.1), (1.2), (1.3). We also recall here the classical papers [4], [5] and [6].

Theorem 1 in reference [1] is proved as an application of a more abstract result, Theorem 2 in the same reference (see 2.1 below). In the sequel we show that this last theorem still applies if we replace the adherence (or non-slip) boundary condition (1.3) by the Navier (slip) boundary condition (1.7) or by the mixed boundary condition (1.8). Actually, the above theorem yields the same kind of results under a variety of boundary conditions, as easily verified. To fix ideas we consider here the slip boundary condition, due to its importance in many theoretical and applied problems. Let us recall this boundary condition. Denote by

$$T = -p\,I + \nu(\nabla v + \nabla v^T)$$

the stress tensor and by $\underline{t} = T \cdot \underline{n}$ the stress vector. Hence, with obvious notation,

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right), \tag{1.4}$$

$$t_i = \sum_{k=1}^{3} T_{ik} n_k. \tag{1.5}$$

We also define the linear operators $v_\tau = v - (v \cdot \underline{n})\,\underline{n}$ (the tangential component of v) and the tangential component of $\underline{t}$

$$\underline{\tau}(v) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.6}$$

Note that $\underline{\tau}(v)$ is independent of the pressure p.