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HYPOELLIPTICITY OF NONLINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Xu Chaojiang
Dept. of Math., Wuhan univ.
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1. Introduction

In order to study nonlinear PDE, the theory of paradifferential operator was introduced by J. M. Bony in [2]. Using this theory, the propagation and interaction of singularities of the solution of non-linear hyperbolic equations were studied by [1], [2] and [3]. This paper will use Bony's theory on the hypoellipticity of non-linear partial differential equations, an abstract of main results of this paper have been published in [8].

Consider the following nonlinear second order partial differential equations:

$$F(x,\ u,\ \nabla u,\ \nabla^2 u)=0 \tag{1.1}$$

where $x\in\Omega$, $\Omega\subset R^n$ is an open set; F is a real valued C^∞ function of real variables. Given a real function $u\in C^\rho_{loc}(\Omega)$, $\rho\geq 4$; we define

$$L=\sum_{j,k=1}^{n}a_{kj}(x)\,\partial_j\partial_k+\sum_{j=1}^{n}b_j(x)\,\partial_j+c(x) \tag{1.2}$$

which is the linearized operator associated with the equation (1.1) for u; where $a_{jk}=a_{kj}\ (j,k=1,2,\ldots,n)$

$$\begin{cases} a_{jk}(x)=\dfrac{\partial F}{\partial u_{jk}}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \\ b_j(x)=\dfrac{\partial F}{\partial u_j}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \qquad j,\ k=1,\ 2,\ \ldots,\ n \\ c(x)=\dfrac{\partial F}{\partial u}(x,\ u(x),\ \nabla u(x),\ \nabla^2 u(x)) \end{cases} \tag{1.3}$$

are all real functions in $C^{\rho-2}_{loc}$. Let us first give the following definition:

Definition 1.1. *The linear operator (1.2) is said to be subelliptic, if $(a_{jk}(x))\geq 0$ for any $x\in\Omega$; and for every compact subset $K\subset\Omega$, there exist constants $\varepsilon>0$, $C>0$, such that for all $\varphi\in C_0^\infty(K)$, the subelliptic estimate:*

$$\|\varphi\|^2_{\varepsilon}\leq C\{|\langle L\varphi,\ \varphi\rangle|+\|\varphi\|^2_0\} \tag{1.4}$$

holds.

Our main theorem is as follows:

Theorem 1.2. *Let $u\in C^\rho_{loc}(\Omega)$, $\rho\geq 4$ be a real solution of equation (1.1). If the linearized operator defined by (1.2) is subelliptic, then the solution $u\in C^\infty(\Omega)$.*

If L is a self-adjoint operator, and the subelliptic index ε is independent of K. Then the consequence of theorem 1.2 is still true if we only suppose $\rho>4-2\varepsilon$. Now it remains to find the sufficient conditions for operator L to be subelliptic. First, if L is elliptic, it is also subelliptic, and $\varepsilon=1$ in this case; this is a classical result. Secondly if operator is degenerate, of course we will consider the so-called Hörmander conditions and Oleinik-Radkevic conditions (see[5], [7]) respectively.

For general operator (1.2), Let

$$\begin{cases} g_j(x, \xi) = \sum_{k=1}^{n} a_{jk}(x)(i\xi_k), & j = 1, \ldots, n \\ g_{n+l}(x, \xi) = |\xi|^{-1} \sum_{j,k=1}^{n} \frac{\partial a_{jk}(x)}{\partial x_l} \xi_j \xi_k, & l = 1, \ldots, n \\ g_0(x, \xi) = \sum_{j=1}^{n} \left(b_j(x) - \sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial x_k}(x)\right)(i\xi_j) \end{cases} \tag{1.5}$$

Then functions $g_j(x, \xi)$ $(0 \leq j \leq 2n)$ are homogeneous of degree 1 and C^∞ in variable $\xi \neq 0$, and $C^{\rho-3}_{loc}$ in variable x. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $0 \leq \alpha_j \leq 2n$ be a multi-index, denote $|\alpha| = k$. If $\rho - 3 \geq |\alpha|$, we define

$$g_\alpha(x, \xi) = (-1)^{k-1}\{g_{\alpha_1}, \ldots, \{g_{\alpha_{k-1}}, g_{\alpha_k}\}, \ldots\} \tag{1.6}$$

to be the Poisson multi-bracket. Then function g_α is a homogeneous degree 1 and C^∞ in $\xi \neq 0$, and $C^{\rho-k-2}_{loc}$ in x. We have:

Theorem 1.3. *Let $u \in C^\rho_{loc}(\Omega)$ be a real solution of equation (1.1); if there exists positive integer p, such that $\rho \geq p + 3$, and the linearized operator (1.2) satisfies:*
(i) $(a_{jk}(x)) \geq 0$ for all $x \in \Omega$.
(ii) For any compact subset $K \subset \Omega$, there exists $C > 0$, such that:

$$\sum_{|\alpha| \leq p} |g_\alpha(x, \xi)|^2 \geq C|\xi|^2 \quad \text{for all } (x, \xi) \in K \times R^n,\ |\xi| \geq R > 0 \tag{1.7}$$

Then the linearized operator L is subelliptic, i. e. $u \in C^\infty$.

If (1.1) is a quasi-linear equation, i. e.

$$\sum_{j=1}^{n} X_j^2 u + X_0 u + f(x, u) = 0 \tag{1.8}$$

where $X_j = \sum_{k=1}^{n} a_{kj}(x, u)\partial_k$, $j = 0, 1, \ldots, m$; a_{kj} and f are all C^∞ real valued functions of real variables. Then replacing $g_j(x, \xi)$ above by

$$X_j(x, \xi) = \sum_{k=1}^{n} a_{kj}(x, u(x))(i\xi_k), \qquad j = 0, 1, \ldots, m$$

theorem 1.3 still holds, under the condition $\rho \geq \max\{2, p\}$.

In theorem 1.2, we need the solution u to be at least C^4. This condition can hardly be improved when (1.1) is a general non-linear and genuinely degenerate equation. C. Zuily[12] proved that there is a solution for a class of degenerate Monge-Ampère equations, which belongs to $C^{2+\varepsilon}$, but not to C^3. In theorem 1.3, more smoothness for the solution u is required. This is because that we need the coefficients of operator to be smooth enough under our assumption, for the Poisson brackets to be definable. In order to improve the condition in theorem 1.3, we introduced the so-called Fefferman-Phong condition in [9], which is kind of geometric subelliptic condition. In [9], we need only $u \in C^4$. Because of the subelliptic conditions in the preceding theorems are all given on a linearized operator of solution u, which must be dependent on u. In [12], for Monge-Ampère equation $\det(u_{ij})(x) = \psi(x)$, C. Zuily gave a condition on function $\psi(x)$, such that its linearized operator satisfies the condition in theorem 1.3, this means the condition in theorem 1.3 may be independent of solution u under some cases. On the other hand, we studied the boundary value problem for a class of non-linear equation (1.1) in [11]; and in [10], higher order equation was discussed.

The plan of this paper is as follows: In Section 2, we will prove the so-called paralinearization theorem of equation (1.1). Section 3 will give the proofs of theorem 1.2 and 1.3. Finally some degenerate cases of theorem 1.3 will be discussed in Section 4.

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