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INITIAL VALUE PROBLEMS FOR NONLINEAR HEAT EQUATIONS

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1. Introduction

In this paper we deal with the global existence and uniqueness of classical solutions to the following initial value problem for nonlinear heat equations

$$\begin{cases} u_t - \Delta u = F(u,\ D_x u,\ D_x^2 u) & (t,\ x) \in R_+ \times R^n \quad (1 \cdot 1) \\ t = 0 : u = \varphi(x) & x \in R^n \quad (1 \cdot 2) \end{cases}$$

where

$$D_x u = (u_{x_1},\ \dots,\ u_{x_n}), \qquad D_x^2 u = (u_{x_i x_j};\ i,\ j = 1,\ \dots,\ n) \quad (1 \cdot 3)$$

and

$$\Delta u = \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right) u \quad (1 \cdot 4)$$

Let

$$\hat{\lambda} = (\lambda;\ (\lambda_i),\ i = 1,\ \dots,\ n;\ (\lambda_{ij}),\ i,\ j = 1,\ \dots,\ n) \quad (1 \cdot 5)$$

Suppose that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1. 1) is suitably smooth and

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}) \quad (1 \cdot 6)$$

where α is an integer ≥ 1.

Based on the Nash-Moser-Hormander iteration scheme, S. Klainerman ([1]) first proved the following result in 1982: If

$$\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right) < \frac{n}{2} \quad (1 \cdot 7)$$

then problem (1. 1) - (1. 2) admits a unique global classical solution on $t \geq 0$, provided that the initial data are small. One year later, S. Klainerman and G. Ponce ([2]) reproved the same result, just using the continuation method of local solutions instead of the Nash-Moser-Hormander iteration.

Observing that for the solution to the heat equation, not only its L^∞-norm but also its L^2-norm decay as $t \to +\infty$, Zheng and Chen ([3]) and G. Ponce ([4]) improved almost at the same time the preceding result by replacing hypothesis (1. 7) with

$$\frac{1}{\alpha} < \frac{n}{2} \quad (1 \cdot 8)$$

To get this improvement, the former still adopted the Nash-Moser-Hormander scheme while the latter used the continuation method of local solutions.

It must be pointed out that in general hypothesis (1. 8) is necessary. As a matter of fact, for the following initial value problem

$$\begin{cases} u_t - \Delta u = u^{1+\alpha}, & (t,\ x) \in R_+ \times R^n \quad (1 \cdot 9) \\ t = 0:\ u = \varphi(x), & x \in R^n \quad (1 \cdot 10) \end{cases}$$

H. Fujita ([5]) and F. B. Weissler ([6]) have proved that if

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$$\frac{1}{\alpha} \geq \frac{n}{2} \qquad (1 \cdot 11)$$

then the classical solution may blow up in a finite time even for sufficiently small initial data.

Moreover, in the case that the nonlinear term F in (1. 1) does not explicitly depend on u : $F = F(D_x u, D_x^2 u)$, without any limitation on the dimension $n \geq 1$, Zheng ([7]) has used once again the Nash-Moser-Hormander scheme to get the global existence of classical solutions for small initial data

In this paper, we give a simple proof to the preceding results, which avoids the use of either the Nash-Moser-Hormander technique or the existence of local solutions. Only based on the decay estimates of solutions to the linear homogeneous heat equation and the energy estimates of solutions to linear inhomogeneous heat equations, we can directly obtain the global existence of classical solutions and some more precise asymptotic behaviors of solutions as $t \to +\infty$. For this purpose, all we have to do is to introduce a function space reflecting simultaneously both the properties of decay and the energy estimates of solutions to corresponding linear problems, and to use the ordinary contraction mapping principle in this space to prove for small initial data the global convergence of the sequence of approximate solutions given by an usual iteration. The method mentioned above can be also systematically used to other nonlinear evolution equations.

2. Preliminaries

Consider the initial value problem for inhomogeneous heat equations

$$\begin{cases} u_t - \Delta u = F(t, x), \quad (t, x) \in R_+ \times R^n & (2 \cdot 1) \\ t = 0 : \quad u = \varphi(x), \quad x \in R^n & (2 \cdot 2) \end{cases}$$

by means of Galerkin' s method we can get

Lemma 2. 1. *For any given $T > 0$, if*

$$\varphi \in H^{s+1}(R^n), \quad F \in L^2(0, T; H^s(R^n)), \qquad (2 \cdot 3)$$

where s is an integer ≥ 0, then problem (2. 1) - (2. 2) admits a unique solution $u = u(t, x)$ satisfying

$$u \in L^2(0, T; H^{s+2}(R^n)) \qquad (2 \cdot 4)$$

$$u_t \in L^2(0, T; H^s(R^n)) \qquad (2 \cdot 5)$$

and

$$\int_0^T \sum_{|k|=2} \| D_x^k u(t, \cdot) \|^2_{H^s(R^n)} dt \leq C_0 \left(\| \varphi \|^2_{H^{s+1}(R^n)} + \int_0^T \| F(t, \cdot) \|^2_{H^s(R^n)} dt \right) \qquad (2 \cdot 6)$$

where C_0 is a positive constant independent of T, $k = (k_1, \ldots, k_n)$ is a multi-index,

$$|k| = k_1 + \ldots + k_n \qquad (2 \cdot 7)$$

and

$$D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \qquad (2 \cdot 8)$$

Corollary 2. 1. *By (2. 4) - (2. 5), we have, with eventual modification on a set of measure zero on $[0, T]$,*

$$u \in C([0, T]; H^{s+1}(R^n)) \qquad (2 \cdot 9)$$

Then, if we suppose furthermore that

$$F \in C([0, T]; H^{s-1}(R^n)) \qquad (2 \cdot 10)$$

by equation (2. 1) we have, with eventual modification on a set of; measure zero on $[0, T]$,

$$u_t \in C([0, T]; H^{s-1}(R^n)) \qquad (2 \cdot 11)$$

We turn now to the decay estimates of solutions to initial value problems for the homogeneous heat equation