J. Partial Differential Equations
Vol. 1 № 1 (1988), 13—20

A NOTE ON PRESCRIBED GAUSSIAN CURVATURE ON S^2 *

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Received June 11, 1987

1. Introduction and Main Results

Given $R(x) \in C^2(S^2)$ where $S^2 = \{x \in R^3 \,|\, |x| = 1\}$, we want to find a condition on $R(x)$ so that there exists a metric g on S^2 with scalar curvature (i. e. twice the Gaussian curvature) $R(x)$, which is pointwise conformal to the standard metric g_0, so $g = e^u g_0$ for some function u.

This problem is equivalent to the existence of a solution of Eq. (cf. [1])

$$\Delta u(x) - 2 + R(x) e^{u(x)} = 0 \qquad x \in S^2 \tag{1.1}$$

where we use the sign convention for Laplacian Δ so that $\Delta u = u_{xx} + u_{yy}$ on flat R^2.

For known results of this interesting problem, confer [1] - [13]. In this paper we prove

Theorem 1. 1. *Assume that $R(x) \in C^2(S^2)$ satisfies*

i) $\exists$ *a curve $\Gamma \in C([0,1], S^2)$, $\Gamma(0) = a \neq b = \Gamma(1)$, $0 < R(b) \leq R(a)$, $b \in S^2$ is a nondegenerate local maximum point of $R(x)$.*

ii) $\min_{x \in \Gamma} R(x) = m < R(b)$ *and* $\forall\ x \in \Gamma \cap R^{-1}(m)$ *either* $\nabla R(x) \neq \vec{0}$ *or* $\nabla R(x) = \vec{0}$, $\Delta R(x) > 0$.

iii) *There is no critical point of $R(x)$ on $R^{-1}(m, R(b))$ except a finite number of nondegenerate local maximum points.*

Then Eq. (1. 1) *has a solution.*

Remark 1. 1. If $\min_{x \in \Gamma} R(x) \leq 0$, assumption ii) can be omitted and assume iii) on $R^{-1}(0, R(b))$, then Theorem 1. 1 remains true.

Remark 1. 2. Notice that Theorem 1. 1 permits $R(b) < R(a) < \max_{x \in S^2} R(x)$, $a \in S^2$ need not be a critical point of $R(x)$, $R(x)$ can be arbitrary on $S^2 \backslash$ a neighborhood of Γ provided iii) holds.

To solve Eq. (1. 1), we look for a critical point of

$$J(u) \triangleq \frac{1}{2}\int_{S^2} |\nabla u|^2 + 2\int_{S^2} u - 8\pi \log \int_{S^2} Re^u \triangleq I(u) - 8\pi \log \int_{S^2} Re^u$$

defined on $H \triangleq \{u \in H^1(S^2) \,|\, \int_{S^2} Re^u > 0\}$. If $J'(u_0) = 0$, then $u = u_0 +$ some constant C is a solution of Eq. (1. 1).

Set $B_r \triangleq \{x \in R^3 \,|\, |x| < r\}$ and $B_1 \triangleq B$. Define $P(u) \triangleq \int_{S^2} xe^u / \int_{S^2} e^u \in B$, $\forall\ u \in H^1(S^2)$. Throughout this paper we assume $R(x) \in C^2(S^2)$. It is worth while noticing the function $m(x) \triangleq \inf_{u \in H,\, P(u) = x} J(u)$, $x \in B$. In section 2 we prove:

* The project supported by National Natural Science Foundation of China

Lemma 2. 1. *If* $R(x)>0$, $\forall x\in S^2$, *then* $m(x)\in C(\bar{B})\cap C^{1-0}(B)$ *and* $m(x)=-8\pi\log(4\pi R(x))$, $\forall x\in S^2$.

In section 3 we prove the following inequality:

Lemma 3. 1. *Suppose that* $b\in S^2$ *is a nondegenerate local maximum point of* $R(x)$, $R(b)>0$, *then there exists* $\delta>0$ *depending on* R *such that* $\forall\ 0<\varepsilon\le\delta$, $\exists\ 0<\mu=\mu(R,\delta,\varepsilon)<4\pi$ *so that the following inequality holds:*

$$\int_{S^2}R(x)e^{u(x)}\le\mu R(b)\exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla u|^2+\frac{1}{4\pi}\int_{S^2}u\right),$$

$$\forall u\in H^1(S^2)\quad with\quad \varepsilon\le|P(u)-b|\le\delta \tag{1. 2}$$

In section 4 we prove Theorem 1.1 using Lemma 2. 1, 3. 1 and minimax argument on H.

2. Function $m(x)$ on Unit Ball $\bar{B}$

In what follows we denote various constants by the same C. Set

$$\varphi_{\lambda y}(x)=\log\frac{1-\lambda^2}{(1-\lambda\cos d(x,y))^2},\qquad x,\ y\in S^2,\qquad 0\le\lambda<1$$

where $d(x,y)$ is the distance on (S^2,g_0) between two points x, y, then (cf. [6]) $u(x)=\varphi_{\lambda y}(x)$ satisfies Eq. (1. 1) with $R(x)=2$.

$$\int_{S^2}\exp(\varphi_{\lambda y}(x))=4\pi,\qquad I(\varphi_{\lambda y}(x))=0 \tag{2. 1}$$

Direct computation shows

$$P(\varphi_{\lambda y})=C(\lambda)\,y\in B,\ C(\lambda)=\frac{1}{\lambda}+\frac{1}{2}\left(\frac{1}{\lambda^2}-1\right)\log\frac{1-\lambda}{1+\lambda} \tag{2. 2}$$

and there is a homeomorphism $h:B\to B$: $\forall\ \lambda y\in B$, $(\lambda,y)\in[0,1)\times S^2$, $h(\lambda y)\triangleq P(\varphi_{\lambda y})$.

Proof of Lemma 2. 1. 1° $J(u)$ is bounded below (cf. [10]) and $J(u)=J(u+C)$ $\forall u\in H^1(S^2)$, $C\in\mathbf{R}$. For fixed $x_0\in B$ choose a minimizing sequence $\{u_i\}\subset H$, $\int_{S^2}u_i=0$, $P(u_i)=x_0$, $J(u_i)\to m(x_0)$. By Aubin [2 Theorem 6], we have

$$\int_{S^2}e^{u_i}\le C\exp\left(\frac{1}{24\pi}\int_{S^2}|\nabla u_i|^2\right) \tag{2. 3}$$

C is independent of i. From (2. 3) and $J(u_i)\le C$ we derive $\|u_i\|_{H^1}\le C$. We can extract a subsequence, still denoted by $\{u_i\}$, such that $u_i\rightharpoonup u_0$ ($H^1(S^2)$). Since $u\in H^1: u\to e^u\in L^1$ is compact (cf. [1 Theorem 2. 46]) and J is weakly lower semicontinuous on H, we get $J(u_0)=m(x_0)$, $P(u_0)=x_0$, i. e. $\inf\limits_{u\in H,\,P(u)=x_0}J(u)=m(x_0)$ is attained by u_0.

2° We prove that $m(x)\in C(B)$. Suppose that $J(u_i)=m(x_i)$, $P(u_i)=x_i\to x\in B$, $\int_{S^2}u_i=0$, using $\varphi_{\lambda y}(x)$ it is easy to see that we can assume $J(u_i)\le C$, again (2. 3) holds, the same reasoning as in 1° shows $\varliminf\limits_{x_i\to x}m(x_i)\ge m(x)$. On the other hand, if $J(u_0)=m(x_0)$, $P(u_0)=x_0$, set $P(u)=p=(p_1,p_2,p_3)$, by definition $\int_{S^2}(x-p)e^u=\vec{0}$, using implicit function theorem we see that there exists a neighborhood U of u_0 in $H^1(S^2)$ such that (v,p) is a coordinate system of U, where v is some subspace of $H^1(S^2)$ with codimension 3. Noticing the continuity of J at $u_0\in H$, we obtain $\varlimsup\limits_{x_i\to x_0}m(x_i)\le m(x_0)$, hence $m(x)\in C(B)$.