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ON PROPERTIES OF SOME OPERATORS IN DOUGLIS ALGEBRA AND THEIR APPLICATION TO PDE

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1. Introduction

Let e and i be two elements generation Douglis algebra[1], which are subject to the following multiplication rules:

$$i^2 = -1,\ ie = ei,\ e^{r+1} = 0,\ e^0 = 1$$

where r is a positive integer.

Definition 1. *We call a hypercomplex value if* $a = \sum_{k=0}^{r} a_k e^k$, *where* a_k $(k=0,\cdots,r)$ *are complex numbers*, a_0 *is called the complex part of* a. *Set* $\bar{a} = \sum_{k=0}^{r} \bar{a}_k e^k$, $|a| = \sum_{k=0}^{r} |a_k|$.

It is easy to know that $|ab| \le |a|\ |b|$, $a\bar{a}$ *is a real hypercomplex value and* $a\bar{a} \ne |a|^2$.

Let $D = \partial_{\bar{z}} + q(z)\partial_z$ be a differential operator, here $q(z)$ is known nilpotent function.

Definition 2. *A hypercomplex function* $w \in C^1(G)$ *is called hyperanalytic if it is a solution of* $Dw = 0$.

A hypercomplex function $w \in C^1(G)$ *is called generalized hyperanalytic function if it is a solution of* $Dw + Aw + B\bar{w} = 0$.

A. Douglis[1], R. P. Gilbert[2],[3], G. Hile[4], H. Begehr[5],[6] and Hou Zongyi[7],[8] have discussed properties of hyperanalytic and generalized hyperanalytic function and their boundary value problem.

Definition 3. *A hypercomplex function* $t(z)$ *is called a generating solution of the operator* D *if*

1) $t(z)$ *has the form* $t(z) = z + \sum_{k=1}^{r} t_k(z)e^k \triangleq z + T(z)$,

2) $T \in B^1(C)$ *and*

3) $Dt(z) = 0$ *in* C.

By

$$\frac{1}{t(\zeta) - t(z)} = \sum_{k=0}^{r} (-1)^k \frac{\Delta(\zeta, z)^k}{(\zeta - z)^{k+1}} \qquad (1\cdot 1)$$

where $\Delta(\zeta, z) = T(\zeta) - T(z)$, *we can get*

$$\left|\frac{1}{t(\zeta) - t(z)}\right| \le \frac{M}{|\zeta - z|},\ \zeta \ne z \qquad (1\cdot 2)$$

where M *is a constant.*

In this paper we deal with some operators in a Douglis algebra and their application to PDE.

R. P. Gilbert[3] introduced Pompieu operator $J_G f = -\frac{1}{\pi}\iint_G \frac{t_\zeta f(\zeta)\, d\sigma_\zeta}{t(\zeta) - t(z)}$ and discussed differential property of J_G, he obtained

$$DJ_G f = f \qquad (1 \cdot 3)$$

and then he investigated a series of properties of J_G, but he could not study operator Π, because the definition of operator J_G is not reasonable.

Now we introduce the differential operators

$$\partial = \alpha(z)\partial_{\bar z} + \beta(z)\partial_z,\ \bar\partial = \overline{\beta(z)}\partial_{\bar z} + \overline{\alpha(z)}\partial z \qquad (1 \cdot 4)$$

where

$$\alpha(z) = -\frac{\overline{t_{\bar z}}}{t_z\overline{t_z} - t_{\bar z}\overline{t_{\bar z}}},\ \beta(z) = \frac{\overline{t_z}}{t_z\overline{t_z} - t_{\bar z}\overline{t_{\bar z}}} \qquad (1 \cdot 5)$$

obviously we have

$$\partial t(z) = 1, \qquad \partial\overline{t(z)} = 0 \qquad (1 \cdot 6)$$

and we also introduce the integral operators.

$$\begin{cases} Tf = -\dfrac{1}{\pi}\displaystyle\iint_G \dfrac{t_\zeta \overline{Dt(\zeta)} f(\zeta) d\sigma_\zeta}{t(\zeta) - t(z)}, \qquad \Pi^* f = -\dfrac{1}{\pi}\displaystyle\iint_G \dfrac{t_\zeta \overline{Dt(\zeta)} f(\zeta) d\sigma_\zeta}{(t(\zeta) - t(z))^2} \\ \Pi f = (\Pi^* - \sigma) f \end{cases} \qquad (1 \cdot 7)$$

where $\sigma = \dfrac{\overline{t_{\bar z}}}{t_z}$. Operator T is different from operator J, since the integrand has weight $\overline{Dt(\zeta)}$ and operator Π is new.

2. Differential Properties of Operator T

In this section we discuss differential properties of operator T in $C_a^m(\bar G)$ and $L_p(\bar G)$.

Theorem 2.1. *Let $G \in C_a^{m+1}$, $f(z) \in C_a^m(\bar G)$, $q \in B^{0,a}(C)$, $0 < a < 1$, $m \geq 0$, then*

1) $T_G f \in C_a^{m+1}(\bar G)$, T_G is a totally continuous operator in $C_a^m(\bar G)$,

2) $\bar\partial T_G f = f$, $\partial T_G f = \Pi f$,

the integral of operator Π is in the Cauchy principle value sense and $\Pi f \in C_a^m(\bar G)$.

Lemma 2.1. *Let G be a bounded domain and ∂G a piecewise smoothly closed curve, $w \in C^1(G)$, it turns out*

$$\iint_G t_\zeta \overline{Dt(\zeta)}\partial w d\sigma_\zeta = -\frac{1}{2i}\int_{\partial G} w d\overline{t(\zeta)} \qquad (2 \cdot 1)$$

$$\frac{1}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)} d\sigma_\zeta}{(t(\zeta) - t(z))^2} = \frac{1}{2\pi i}\int_{\partial G} \frac{\overline{t(\zeta)} dt(\zeta)}{(t(\zeta) - t(z))^2} - \sigma(z) \qquad (2 \cdot 2)$$

where the integral in the left of (2,2) is in the Cauchy principle value sense.

Proof. By applying Green formula, Pompieu formula[3] and properties of $t(z)$, this lemma holds obviously.

Now we return to the proof of theorem 2.1.

Proof. we assume that $m = 0$ at first.

$$\Pi^* f = -\frac{1}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z))}{(t(\zeta) - t(z))^2} d\sigma_\zeta - \frac{f(z)}{\pi}\iint_G \frac{t_\zeta \overline{Dt(\zeta)} d\sigma_\zeta}{(t(\zeta) - t(z))^2} \qquad (2 \cdot 3)$$

when $f \in C_a(\bar G)$, the first integral is a weak singular integral. By use of lemma 2.1 the second integral is in the Cauchy principle value sense.

We set

$$g(z) = \Pi f,\ \Delta_1 = t(\zeta) - t(z_1),\ \Delta_2 = t(\zeta) - t(z),\ \Delta_0 = t(z_1) - t(z)$$

thus

$$g(z_1) - g(z) = -\frac{1}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)} f(\zeta)\left[\frac{1}{\Delta_1^2} - \frac{1}{\Delta_2^2}\right] d\sigma_\zeta - f(z_1)\sigma(z_1) + f(z)\sigma(z)$$

$$= -\frac{\Delta_0}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z_1))\frac{d\sigma_\zeta}{\Delta_1^2\Delta_2} - \frac{\Delta_0}{\pi}\iint_G t_\zeta \overline{Dt(\zeta)}(f(\zeta) - f(z))\frac{d\sigma_\zeta}{\Delta_1\Delta_2^2}$$