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ON A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF MIXED TYPE *

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Partial differential equations of mixed type has been a very active topic since F. Tricomi' s pioneering work on the equation

$$yu_{xx} + u_{yy} = 0 \tag{1}$$

which bears his name. This is mainly due to the significant role it plays in the theory of transonic flow. It also appears in various fields, for instance in the theory of plasticity and the theory of deformation of surfaces, just to name a few of them. There is another type of partial differential equation of mixed type. M. Cibrario [1] considered the general second order equation

$$\begin{aligned} &A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} \\ &+ D(x, y)u_x + E(x, y)u_y + F(x, y)u = 0 \end{aligned} \tag{2}$$

where the coefficients are real analytic functions of real variables (x, y) and the discriminant $\Delta(x, y) = B^2 - AC$ may change sign across the type-changing curve $\Gamma : \Delta = 0$ and is of mixed type there. She proved that equation (1) can always be reduced to either of the following forms

$$y^{2m+1}u_{xx} + u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \tag{3_1}$$

$$u_{xx} + y^{2m+1}u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \tag{3_2}$$

Thus, Tricomi's equation is only the simplest model of (3_1) where Γ is not characteristic. Equations of the form (3_2) is also of considerable interest. The earliest example is

$$u_{xx} + yu_{yy} + au_y = 0 \qquad a = \text{const.} \tag{4}$$

which has been studied by I. P. Carol' [2]. Let Ω be a domain in (x, y) plane such that $\Omega \cap \{y \neq 0\} \neq \varnothing$. $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, Γ_1 is an arc lying in $y \geq 0$ with end point A and B on $y = 0$, Γ_2 and Γ_3 are characteristics of (4) in $y \leq 0$ through A and B respectively. When $a < 0$, Carol' proved that the Dirichlet problem (problem M) for (4) is well-posed, while for $a > 0$, boundary value can be assigned on Γ_1 (problem E).

Equations of the type (3_2) also appear in gas dynamics (for instance, conic flow) [3] where we are required to solve the Busemann equation

$$(1 - x^2)u_{xx} - 2xyu_{xy} + (1 - y^2)u_{yy} + 2axu_x + 2ayu_y - a(a+1)u = 0 \tag{5}$$

The unit circle $x^2 + y^2 = 1$ is the type-changing curve and also a characteristic curve for the equation (5), which in polar coordinates can be written as

$$(1 - r^2)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial \theta^2} + \left(\frac{1}{r} + 2ar\right)\frac{\partial u}{\partial r} - a(a+1)u = 0 \tag{5_1}$$

Near the type-changing curve $r = 1$, (5_1) becomes asymptotically

$$\rho\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} - \left(\frac{1}{2} + a\right)\frac{\partial u}{\partial \rho} - a(a+1)u = 0, \qquad \rho = 1 - r \tag{5_2}$$

Gu Chao-hao proved in [4] that the Dirichlet problem is well-posed when $a > \frac{1}{2}$, and when $a < \frac{1}{2}$ boundary value can be assigned only on that part of the boundary inside the elliptic domain $x^2 + y^2 < 1$.

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Equation of the type (3_2) also appears in magneto-hydrodynamics, see Seebass [5].

In this paper, we consider the second order linear equation

$$u_{xx} + yu_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0, \qquad c \leq 0 \tag{6}$$

(i. e. (3_2) with $m = 0$) in a region $\Omega = \Omega_+ \bigcup \Omega_-$, $\Omega_\pm = \Omega \bigcap \{y \geq, \leq 0\}$. We assume a, b, c are analytic functions of the real variables x and y. We also assume

$$b(x, 0) = b_0 = \text{const.} \tag{7}$$

Our main idea is that, any solution of (6) is "glued up" from 2 solutions each in Ω_+ and Ω_-. The smoothness of the solution is determined by the constant b_0 in (7). Different requirements on smoothness lead to corresponding boundary value problem. More precisely, our main results are: first, we prove that all solutions of (6) admit an asymptotic expansion

$$u(x, y) \sim \sum_{n=0}^{\infty} a_n(x) y^n + y^\lambda \sum_{n=0}^{\infty} b_n(x) y^n \qquad \lambda = 1 - b_0 \tag{8}$$

near $y = 0$ (but $y \neq 0$). Next, we prove that all solutions in Ω_+ or Ω_- can be extended "analytically" into Ω_- or Ω_+ respectively. From these results we can give well-posed boundary value problems.

Equation (6) is a Fuchsian type partial differential equation. From (8) it is seen that $y = 0$ is a singularity of the solution, while for equation (3_1) there is no solution with remarkable singularity on $y = 0$. This would help to explain the difference between (3_1) and (3_2).

From the condition (7), we have

$$b(x, y) = b_0 + yb_1(x, y)$$

Introducing a new unknown function $v(x, y)$

$$v(x, y) = u(x, y)\exp\left[\frac{1}{2}\int a(x, 0)\,dx + \frac{1}{2}yb_1(x, 0)\right]$$

equation (6) becomes an equation in $v(x, y)$:

$$\begin{aligned} v_{xx} + yv_{yy} &+ [a(x, y) - a(x, 0) - yb_1'(x, 0)]v_x \\ &+ [b(x, y) - yb_1(x, 0)]v_y + \tilde{c}v = 0 \end{aligned}$$

where $\tilde{c}(x, y)$ is an analytic function in (x, y) near $y = 0$. Since

$$a(x, y) - a(x, 0) - yb_1'(x, 0) = ya_1(x, y)$$

$$b(x, y) - yb_1(x, 0) = b_0 + y[b_1(x, y) - b_1(x, 0)] = b_0 + y^2 b_2(x, y)$$

hence, without losing generality, we may assume that the coefficients of (6) $a(x, y)$, $b(x, y)$ are of the form

$$\begin{aligned} a(x, y) &= ya_1(x, y) \\ b(x, y) &= b_0 + y^2 b_1(x, y) \end{aligned} \tag{9}$$

Using characteristic variables

$$\xi = x + 2(-y)^{\frac{1}{2}}$$

$$\eta = x - 2(-y)^{\frac{1}{2}}$$

(6) can be written as

$$\begin{aligned} u_{\xi\eta} &- \left[\frac{\beta'}{\xi - \eta} + (\xi - \eta)^2 A(\xi, \eta)\right]u_\xi + \\ &+ \left[\frac{\beta}{\xi - \eta} - (\xi - \eta)^2 B(\xi, \eta)\right]u_\eta + C(\xi, \eta)u = 0 \end{aligned} \tag{10}$$

with $\beta = \beta' = -\frac{1}{2} + b_0 = \text{const}$, $A(\xi, \eta)$, $B(\xi, \eta)$ and $C(\xi, \eta)$ analytic in ξ, η near $\xi = \eta$.

In hyperbolic region Ω_- where $y \leq 0$, ξ, η are real variables, while in elliptic region Ω_+ where $y \geq 0$, ξ, η are complex conjugate. But in the following we would treat ξ and η as independent complex variables.

Now we give the following

Definition. *If* $(\xi - \eta)^{-\rho} f(\xi, \eta)$ *is an analytic function of* ξ *and* η *near* $\xi = \eta$, *we say*