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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR THE BELOUSOV-ZHABOTINSKII REACTION DIFFUSION SYSTEM

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Abstract

The present paper characterizes the asymptotic behavior of the timedependent solution of the coupled Belousov-Zhabotinskii reaction diffusion equations in relation to the steady-state solutions of the corresponding boundary value problem. This characterization leads to an explicit relationship among the various physical constants and the boundary and initial functions.

1. Introduction

In some chemical reaction problems, a simplified model for the concentration densities $u \equiv u(t, x)$, $v = v(t, x)$ of two reactants, such as bromous acid and bromide ion, is given by a coupled system of reaction diffusion equations in the form

$$\begin{aligned} u_t - D_1 \nabla^2 u &= u(a - bu - cv) \\ v_t - D_2 \nabla^2 v &= -c_1 uv \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{1.1}$$

Where D_1, D_2, a, b, c and c_1 are positive constants and Ω is the reaction-diffusion medium. The coupled system is often referred to as the Belousov-Zhabotinskii chemical reaction equations and has been given considerable attention in recent years (cf. [1-5, 10]). Much discussion of Eq. (1.1) is devoted to the traveling wave solution in the one-dimensional spatial domain $\Omega \equiv R^1$. When Ω is a general bounded domain in R^n, Eq. (1.1) is supplemented by a boundary condition in the form

$$\begin{aligned} \alpha(x)\, \partial u/\partial \nu + \beta(x)\, u &= 0 \\ \alpha(x)\, \partial v/\partial \nu + \beta(x)\, v &= 0 \end{aligned} \qquad (t > 0,\ x \in \partial\Omega) \tag{1.2}$$

together with the initial condition

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \qquad (x \in \Omega) \tag{1.3}$$

where $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta > 0$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$, and $u_0 \geq 0$, $v_0 \geq 0$ in Ω. It is assumed that the functions in (1.2) (1.3) and the domain Ω are smooth and $\beta(x)$ is not identically zero (see [5] for the case $\beta(x) \equiv 0$).

It has been shown in [5] that for any nonnegative initial function u_0, v_0, problem (1.1) - (1.3) has a unique nonnegative solution (u, v). The aim of this paper is to give a more precise description about the asymptotic behavior of the solution (u, v) in relation to the steady-state solutions of the corresponding boundary-value problem

$$\begin{aligned} -D_1 \nabla^2 u &= u(a - bu - cv) \\ -D_2 \nabla^2 v &= -c_1 uv \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{1.4}$$

$$\begin{aligned} \alpha(x)\, \partial u/\partial \nu + \beta(x)\, u &= 0 \\ \alpha(x)\, \partial v/\partial \nu + \beta(x)\, v &= 0 \end{aligned} \qquad (t > 0,\ x \in \partial\Omega) \tag{1.5}$$

Since problem (1.4) (1.5) has the trivial solution (0, 0) it is interesting to know when it has a nontrivial solution, and whether and when the time dependent solution (u, v) converges to the nontrivial solution. Our main results characterize the asymptotic behavior of the solution (u, v) in terms of the various physical constants in (1.1) as well as the effect of the boundary and initial conditions (1.2) (1.3).

2. The Main Results

The characterization of the existence of a nontrivial steady-state solution and its relation to the time-dependent solution is based on the smallest eigenvalue λ_0 and its corresponding eigenfunction $\varphi(x)$ of the eigenvalue problem

$$\nabla^2\varphi + \lambda\varphi = 0 \quad \text{in } \Omega,\ B[\varphi] = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

where $B[w] \equiv \alpha(x)\,\partial w/\partial\nabla + \beta(x)\,w$ for any function w. It is well-known that $\lambda_0 > 0$ and $\varphi(x) > 0$ in Ω. When $\alpha(x) > 0$ the maximum principle implies that $\varphi(x) > 0$ on $\bar{\Omega}$. We normalize φ so that $\max \varphi(x) = 1$ on $\bar{\Omega}$. The following existence result for the scalar boundary value problem

$$-D_1\nabla^2 U = U(a - bU) \ \text{ in } \Omega,\ B[U] = 0 \ \text{ on } \partial\Omega \tag{2.2}$$

is well-known.

Lemma 2.1. *Problem (2.2) has only the trivial solution $U = 0$ when $a \le \lambda_0 D_1$; and it has a unique positive solution $U_s(x)$ when $a > \lambda_0 D_1$.*

A proof of the above lemma can be found in [9, p. 1174]. Based on the solution U_s of problem (2.2) we state our main results in the following two theorems.

Theorem 1. *The steady-state problem (1.4) (1.5) has only the trivial solution (0, 0) when $a \le \lambda_0 D_1$; and it has exactly two solutions (0, 0) and $(U_s, 0)$ when $a > \lambda_0 D_1$, where U_s is the unique positive solution of (2.2).*

Theorem 2. *Let (u, v) be the nonnegative solution of (1.1) - (1.3) with any $(u_0, v_0) \ge (0, 0)$ and let U_s be the positive solution of problem (2.2). Then*

$$\lim_{t\to\infty} (u(t, x),\ v(t, x)) = (0,\ 0) \tag{2.3}$$

when $a \le \lambda_0 D_1$ or when $u_0(x) \equiv 0$, and

$$\lim_{t\to\infty} (u(t, x),\ v(t, x)) = (U_s(x),\ 0) \tag{2.4}$$

when $a > \lambda_0 D_1$ and $u_0(x) \ge \varepsilon\varphi(x)$, where $\varepsilon > 0$ can be arbitrarily small.

Remark 2.1. When $a > \lambda_0 D_1$ the conclusion in (2.4) also hold for any $U_0(x) \not\equiv 0$ provided that $\alpha(x) > 0$. For in this situation the maximum principle implies that $u(t, x) > 0$ on $R^+ \times \bar{\Omega}$. By considering problem (1.1) - (1.3) with the initial functions $u(t_1, x)$, $v(t_1, x)$ in the domain $[t_1, \infty) \times \Omega$ for a fixed $t_1 > 0$, the requirement $u(t_1, x) \ge \varepsilon\varphi(x)$ for some $\varepsilon > 0$ is clearly satisfied. It follows from the uniqueness property of the solution (u, v) that (2.4) holds.

3. Proof of the Main Theorems

Proof of Theorem 1. Let $(U_s(x), V_s(x))$ be any nonnegative solution of (1.4) (1.5). Multiplying both equations in (1.4) by $\varphi(x)$ and integrating over Ω yield

$$-D_1\int_\Omega \varphi\nabla^2 U_s dx = \int_\Omega \varphi U_s(a - bU_s - cV_s)\,dx$$

$$-D_2\int_\Omega \varphi\nabla^2 V_s dx = -c_1\int_\Omega \varphi U_s V_s dx$$

By applying the Green's theorem and using the boundary condition (1.5) the above equations become

$$\begin{aligned}(\lambda_0 D_1 - a)\int_\Omega \varphi U_s dx &= -\int_\Omega (b\varphi U_s^2 + c\varphi U_s V_s)\,dx \\ \lambda_0 D_2\int_\Omega \varphi V_s dx &= -c_1\int_\Omega \varphi U_s V_s dx \le 0\end{aligned} \tag{3.1}$$

Since $\varphi(x) > 0$ in Ω, the second relation in (3.1) implies that $V_s(x) = 0$ in Ω.