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THE CLASSICAL SOLUTION OF THE PERIODIC STEFAN PROBLEM

Yin Hongming
(Department of Mathematics, Peking University)

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1. Introduction

In this paper, we are concerned with the periodic Stefan problem in the one-dimensional case. This problem is a mathematical model of many practical problems. For example, when a railway is constructed in severely cold district, it is necessary to consider the thickness of frozen soil layer. It is an unknown function and varies with temperature. If we regard the average temperature for every year as to be about the same, i. e. the temperature is approximately a periodic function of the time, then the thickness of the frozen soil layer should also approximately be a periodic function of the time. By the phase-change theory and the Fourier heat conduct law, we can derive a strict mathematical model for the above problem. After standardizing some quantities, we have the following simple statement.

Problem (A): Find a pair of functions $(s(t), u(x,t))$, $s(t): R^1 \to (0, 1)$ in $C^1(R^1)$ and $u(x,t): \quad \bar{Q} = \{(x,t): 0 \le x \le 1, t \in R^1\} \to R^1$ in $C^{2,1}(Q_1) \cup C^{2,1}(Q_2)$ satisfying following equations and data:

$$a_1(u)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad Q_1 = \{(x, t): 0 < x < s(t), \ t \in R^1\} \tag{1.1}$$

$$a_2(u)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad Q_2 = \{(x, t): s(t) < x < 1, \ t \in R^1\} \tag{1.2}$$

$$u(0, t) = f_1(t) > 0 \qquad -\infty < t < +\infty \tag{1.3}$$

$$u(1, t) = f_2(t) < 0 \qquad -\infty < t < +\infty \tag{1.4}$$

$$u(s(t), t) = 0 \qquad -\infty < t < +\infty \tag{1.5}$$

$$\dot{s}(t) = -u_x^-(s(t), t) + u_x^+(s(t), t) \qquad -\infty < t < +\infty \tag{1.6}$$

$$u(x, t+T) = u(x, t) \qquad \text{on } \bar{Q} \tag{1.7}$$

$$s(t+T) = s(t) \qquad -\infty < t < +\infty \tag{1.8}$$

where $\dot{s}(t) = \frac{ds(t)}{dt}$, $u_x^-(s(t), t) = \lim_{x \uparrow s(t)} u_x(x, t)$, $u_x^+(s(t), t) = \lim_{x \downarrow s(t)} u_x(x, t)$ and $f_i(t)$ is given periodic functions with period T $(i = 1, 2)$.

The periodic Stefan problem has been discussed by several authors. In 1980, H. Ishii[7] proved the existence and uniqueness of an almost periodic solution for the one-dimensional two-phase Stefan problem with heat conduct equations. At the same time, M. Stedry and O. Vejvoda[11] by using the implicit function theorem, established similar results for the same equations, but the boundary values were confined to a special class which was obtained by a constant with a sufficiently small perturbation. It is difficult to deal with quasilinear parabolic equations by their methods. In our work, we use a new argument to study the problem with quasilinear parabolic equations. We will prove the existence and uniqueness of the solution of the problem (A). It is worth noticing that the method used here give a direct proof from weak solution of Stefan problem to classical solution (see section 4 for detail) and also can be used to investigate other free boundary problems.

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Throughout this paper, we always assume the following conditions:

(H1): $a_i(\xi) \in C^3(R^1)$ and there exists a positive constant a_0 such that

$$a_i(\xi) > a_0 > 0 \text{ for all } \xi \in R^1, \qquad (i=1,\ 2)$$

(H2): $f_i(t) \in C^3(R^1)$ and $f_i(t+T) = f_i(t)$ for all $t \in R^1$, $\quad (i=1,\ 2)$

Furthermore, there exists a constant $\alpha_0 > 0$ such that

$f_1(t) > \alpha_0 > 0$ and $f_2(t) < -\alpha_0 < 0$, where T is given constant.

2. The Existence of the Weak Solution

In this section, we will change the problem (A) into a weak form and prove the existence of the weak solution. The basic method is the estimation of varies integrals of the solution for a suitable approximation problem.

Before we discuss the problem, we introduce some function spaces.

Let

$$C_T^{m+\alpha}(R) = \{g(t) \in C^{m+\alpha}(R^1) \text{ and } g(t+T) = g(t) \text{ for all } t \in R^1\}$$

$$C_T^{m+\alpha,\,k+\beta}(Q) = \{\varphi(x,\ t) \in C^{m+\alpha,\,k+\beta}(Q) : \varphi(x,\ t+T) = \varphi(x,\ t) \text{ for all } (x,\ t) \in \bar{Q}\}$$

$$\dot{C}_T^{2,1}(\bar{Q}) = \{\varphi(x,\ t) \in C_T^{2,1}(\bar{Q}) : \varphi(0,\ t) = \varphi(1,\ t) = 0 \text{ for all } t \in R^1\}$$

where m and k are non-negative integers and α, β and T are constants ($0 \le \alpha \le 1$, $0 \le \beta \le 1$).

Definition: *A continuous function $u(x,\ t)$ defined on $\bar{Q} = \{(x,\ t) : 0 \le x \le 1,\ t \in R^1\}$ is called a weak solution of the periodic Stefan problem (A), if $u(x,\ t)$ satisfies the integral*