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EQUILIBRIA OF INITIAL-BOUNDARY VALUE PROBLEM FOR $u_t = (u^m)_{xx} + (a - x^2)u$

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1. Introduction

In this paper we consider the following boundary value problem:

$$\begin{cases} u_t = (u^m)_{xx} + (a - x^2)u, & |x| \leq L, \ t \geq 0 \\ u(-L, t) = u(L, t) = 0 & t \geq 0 \\ u(x, 0) = u_0(x) \geq 0 & |x| \leq L \end{cases} \tag{1}$$

where $L > 0$ is a parameter.

The solution of (1) defines a local dynamical system such that $v = u^m \in H^1(-L, L)$. Moreover, the ω -limit set of any bounded orbits is a equilibrium of (1); that is, a solution of

$$(u^m)_{xx} + (a - x^2)u = 0 \tag{2}$$

$$u(\pm L) = 0 \tag{3}$$

The problem arises in certain biological populations (See[7]). Recently, some authors have considered other similar questions, for instance, $f(x, u) = f(x)u$ instead of the $(a - x^2)u$ in this problem, where $f(x)$ is a step function, see[5]. When $f(x, u) = u(1 - u)(u - a)$, [4] gives a complete discussion. The authors of [5] and [4] have used the standard phase-energy method. Here in our case, $f(x, u) = f(x)g(u)$ and $f(x)$ is not a contant or a step function, namely, (2) is nonautonomous. To overcome this difficulty in section 3 we give some new ideas. Using the symmetricity and monotonicity of solution of (2) and (3) reduce (2) and (3) to a equivalent operator equation. And using Schauder fixed point theorem we get the local bifurcation of the nonnegative equilibria of (2) and (3). Finally we use Implicit Function Theorem to obtain some further results. We have proved that for $m \in (1, 3/2)$, the bifurcation is global, and for $m \in [3/2, \infty)$, we give a sufficient condition under which we can extend the local bifurcation. In section 2 we also give some results about the stable properties of the equilibria.

2. Comparison and Local Existence Theorem

Consider the following problem:

$$
(\mathrm{I})\quad
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 \Phi(u)}{\partial x^2} + f(x, u) & |x| \leq L,\ t \geq 0 \\
u(\pm L,\ t) = 0 & t \geq 0 \\
u(x,\ 0) = u_0(x) & |x| \leq L
\end{cases}
$$

where $\Phi(u) = |u|^m \mathrm{sign}u$, $m > 1$, $f(x, u) = (a - x^2) u$, $a > 0$. From P. E. Sacks [1] it follows that (I) has at least a solution $u \in L^\infty(Q_T) \cap C(Q_T)$ for $T > 0$, $Q_T = (-L, L) \times (0, T)$. Crandall & Bresis [2] prove the uniqueness. We also consider the asymptotic behaviour of nonnegative solutions of (I). Therefore we first consider the stationary problem, that is, the following one:

$$
(\mathrm{II})_L
\begin{cases}
(u^m)_{xx} + (a - x^2) u = 0 & |x| \leq L \\
u(\pm L) = 0 & u \geq 0
\end{cases}
$$

Definition. *u is called the positive (upper-, lower-) solution, if $u > 0$ for $x \in \Omega = (-L, L)$, $u \in L^\infty(\Omega)$ and satisfies*

$$\int_{-L}^{L} [u^m \varphi_{xx} + (a - x^2) u\varphi] dx = (\leq, \geq) 0$$

for every $\varphi \in C_0^2(-L, L)$.

Lemma 1. *Suppose that $Z(x) \in L^1(\Omega)$, $W(x) \in L^\infty$, and $W_{xx} + Z = 0$ holds in the sense of distributions. Then $W_x \in L^\infty(\Omega)$.*

Proof. Set $\Omega' = [\delta - L, L - \delta]$, $\delta > 0$, Using mollified operator ρ_h, $h < \delta$, we set $u_h = \rho_h * u$, then $u_h \to u$ in L^1 if $u \in L^1$, as $h \to 0^+$. Since $D^\alpha u_h = (D^\alpha u)_h$ (See[8]), we have $W_{hxx} + Z_h = 0$. This gives $w_{hxx} \in L^1$ and $\| W_{hxx} \|_{L^1} \leq \| Z_h \|_{L^1}$, ess var $W_{hx} \leq \| W_{hxx} \|_{L^1} \leq \| Z_h \|_{L^1} \to \| Z \|_{L^1}$, as $h \to 0$. $W_h \in L^\infty$, and $\| W_h \|_{L^\infty} \leq C_0$, where C_0 independent of the h. So there exists a point $x_0 \in [-L + \delta, L - \delta]$, such that

$$
\begin{aligned}
|W_{hx}(x_0)| &= |[-W_h(-L + \delta) + W_h(L - \delta)]/2(L - \delta)| \\
&\leq 2(C_0 + 1)/(L - \delta) \leq 4(C_0 + 1)/L, \quad (\text{if} \quad \delta < L/2)
\end{aligned}
$$

Then

$$
\begin{aligned}
|W_{hx}| &\leq |W_{hx}(x_0)| + \text{ess var} W_{hx} \\
&\leq 4(C_0 + 1)/L + \| Z \|_{L^1} + 1
\end{aligned}
$$

So in L^p, $1 \leq p < \infty$, there exists a subsequence $\{W_{h_i}\}$, $(W_{h_i})_x \to W_x$ (in L^p). This yields $(W_{h_i})_x \xrightarrow{\text{a. e.}} W_x$. Using $\| W_{hx} \|_{L^\infty} \leq M$, where M is independent of h, we at last get $\| W_x \|_{L^\infty} \leq C$. The proof is completed.

Lemma 2. *If u is a nonnegative solution of $(\mathrm{II})_L$, then $v = u^m \in C^{2+\beta}$, $\beta = \alpha/2 = 1/2m$. so in $\Omega_+ = \{u > 0\}$, $u \in C^2$.*

Proof. Set $v = u^m$, $\alpha = \dfrac{1}{m} \in (0, 1)$, then