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# PARTIAL REGULARITY FOR THE 2-DIMENSIONAL WEIGHTED LANDAU-LIFSHITZ FLOW*

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**Abstract** We consider the partial regularity of weak solutions to the weighted Landau-Lifshitz flow on a 2-dimensional bounded smooth domain by Ginzburg-Landau type approximation. Under the energy smallness condition, we prove the uniform local $C^\infty$ bounds for the approaching solutions. This shows that the approximating solutions are locally uniformly bounded in $C^\infty(\mathrm{Reg}(\{u_\epsilon\})\bigcap(\bar{\Omega}\times R^+))$ which guarantee the smooth convergence in these points. Energy estimates for the approximating equations are used to prove that the singularity set has locally finite two-dimensional parabolic Hausdorff measure and has at most finite points at each fixed time. From the uniform boundedness of approximating solutions in $C^\infty(\mathrm{Reg}(\{u_\epsilon\})\bigcap(\bar{\Omega}\times R^+))$, we then extract a subsequence converging to a global weak solution to the weighted Landau-Lifshitz flow which is in fact regular away from finitely many points.

**Key Words** Landau-Lifshitz equations; Ginzburg-Landau approximations; Hausdorff measure; partial regularity.

**2000 MR Subject Classification** 35J55, 35J60.

**Chinese Library Classification** O175.29.

## 1. Introduction

In this paper, we are concerned with the existence and regularities of global weak solutions to initial and boundary value problem for the weighted Landau-Lifshitz flow

$$\begin{aligned}
&\frac{1}{2}\partial_t u - \frac{1}{2}u \times \partial_t u - \nabla\cdot(a(x)\nabla u) = a(x)|\nabla u|^2 u \quad \text{in} \quad \Omega\times R_+, \qquad (1.1)\\
&u = u_0 \quad \text{on} \quad \Omega\times\{0\}\bigcup\partial\Omega\times R_+,
\end{aligned}$$

where "$\times$" denotes the usual vector product in $R^3$, the domain $\Omega\subset R^2$ is open, bounded and smooth. The initial and boundary data $u_0$ is assumed to be a smooth map into

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the standard sphere $S^2 \subset R^3$. In the classical sense, the equation (1.1) is equivalent to

$$u_t = u \times \nabla \cdot (a(x)\nabla u) - u \times (u \times \nabla \cdot (a(x)\nabla u)).$$

This problem is a special case of magnetization motion equation suggested in 1935 by Landau and Lifshitz, i.e.

$$\frac{\partial S}{\partial t} = \lambda_1 S \times H^e - \lambda_2 S \times (S \times H^e),$$

where $\lambda_2 > 0$ is the Gilbert damping constant, $\lambda_1$ is a constant, $S = (S_1, S_2, S_3)$ is the magnetization vector, and $H^e$ is effective field which can be computed by the formula $H^e := \frac{\partial}{\partial S} e_{\mathrm{mag}}(u)$, $e_{\mathrm{mag}}(u)$ being the total energy. In particular, if we take nonhomogeneous effective magnetic energy as $e_{\mathrm{mag}}(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 dx$, we then obtain (1.1).

If $a(x) \equiv 1$, the equation (1.1) reads as

$$\frac{1}{2}\partial_t u - \frac{1}{2} u \times \partial_t u - \triangle u = |\nabla u|^2 u \quad \text{in} \quad \Omega \times R_+,$$

which has been widely discussed by mathematicians. Early in 1987, Zhou and Guo [1] had obtained the global existence of weak solutions and in 1991 [2], Zhou, Guo and Tan established the existence and uniqueness of smooth solution for 1-D problem. In 1993, for 2-D problem, Guo and Hong [3] found the close relations between this equation and harmonic map heat flow and proved the existence of partially regular solution which was first obtained for harmonic map heat flow by Chen and Struwe [4]. In 1998, also for 2-D problem, Chen Y. Ding S. and Guo B. proved that any weak solution with finite energy is smooth away from finitely many points [5]. For high dimensional problem, we refer to recent results by Liu [6] for the partial regularity of stationary weak solutions, by Ding and Guo [7] for the partial regularity of stationary weak solutions to Landau-Lifshitz-Maxwell equations in 3 dimensions, and by [8] for the partial regularity of weak solutions in 3 and 4 dimensions. We refer also to Paul Harpes' results [9] for the partial regularity of 2-D problem by Ginzburg-Landau approximations.

Concerning the Landau-Lifshitz equation where the coefficient is a function, $a(x) \not\equiv$ constant, there are not many discussions. So far as we know, the only results are the following. In 1999, Ding S., Guo, B. and Su, F [10] obtained the existence of measure-valued solution to the 1-D compressible Heisenburg chain equation

$$\vec{Z}_t = (G(\vec{Z}_x)\vec{Z} \times \vec{Z}_x)_x,$$

where $G(\vec{Z}_x)$ is a matrix function. In the same year, in [11], these authors proved the existence and uniqueness of smooth solution to the 1-D inhomogeneous equation

$$\vec{Z}_t = f(x)\vec{Z} \times \vec{Z}_{xx} + f'(x)\vec{Z} \times \vec{Z}_x.$$

Recently, Lin, J. and Ding, S. extends this problem in [12], where the function $f(x)$ is replaced by $f(x,t)$ and the method to get the estimates is different from that in [11].