J. Partial Diff. Eqs.
20(2007), 80–96 Vol.20 No.1

NEW EXPRESSIONS OF PERIODIC WAVES AND A NOVEL PHENOMENON IN A COMPRESSIBLE HYPERELASTIC ROD*

Liu Zhengrong and Zhang Bengong
(School of Mathematics Science, Center for Nonlinear Science Studies,
South China University of Technology, Guangzhou, Guangdong, 510640, China)
(E-mail: liuzhr@scut.edu.cn)

Dedicated to Prof. Boling Guo for his 70th Birthday

(Received Jul. 24, 2006)

Abstract A In this paper, we employ both bifurcation method of dynamical systems and numerical exploration of differential equations to investigate the periodic waves of a general compressible hyperelastic rod equation

$$u_t + 3uu_x - u_{xxt} - \gamma(2u_x u_{xx} + uu_{xxx}) = 0,$$

with parameter $\gamma < 0$. New expressions including explicit expressions and implicit expressions are obtained. Some previous results are extended. Specially, a new phenomenon is found: when the initial value tends to a certain number, the periodic shock wave suddenly changes into a smooth periodic wave. In dynamical systems, this represents that one of orbits can pass through the singular line. The coherency of numerical simulation and theoretical derivation implies the correctness of our results.

Key Words Hyperelastic rod; bifurcation method; numerical exploration; periodic waves.

2000 MR Subject Classification 34A20, 34C25, 35B65, 58F05, 76B25.

Chinese Library Classification O175.2.

1. Introduction

Many authors have studied nonlinear waves in elastic rods. For instance, Wright [1] considered traveling waves in a rod composed of an incompressible hyperelastic material. Samsonov [2] obtained the so-called double dispersive equation and showed that it has a solitary wave solution. Coleman and Newman [3] derived the one-dimensional rod equation for a general incompressible hyperelastic material. Dai [4] studied disturbances in an initially stretched or compressed rod which is composed of a compressible Mooney-Rivlin material and derived a new type of nonlinear dispersive equation as the model equation which takes the following form:

$$u_t + 3uu_x - u_{xxt} - \gamma(2u_x u_{xx} + uu_{xxx}) = 0, \tag{1.1}$$

---

*Research is supported by the National Natural Science Foundation of China (No.10571062).

where γ is a physical parameter. Dai and Huo [5, 6] used phase portraits of traveling wave system to show that Eq.(1.1) has a variety of travelling waves, including solitary shock waves, solitary waves, periodic shock waves, etc. Also a new phenomenon was found in [5]: a solitary wave can suddenly change into a periodic wave (with finite period). In dynamical systems, this represents that a homoclinic orbit suddenly changes into a closed orbit. Constantin and Strauss [7] proved that the solitary waves of Eq.(1.1) are orbitally stable, which establish that the shape of the waves are stable. Yin [8] studied the Cauchy problem of Eq.(1.1). Liu and Chen [9] got some implicit expressions of the compactons for Eq.(1.1), which include Jacobian elliptic functions. Dai et al [10] provided some theoretical results to deal with singular solutions and obtained an explicit expression of the compactons for Eq.(1.1).

With $\gamma = 1$ in Eq.(1.1), we find Camassa-Holm equation which has been studied successively by many authors; see for instance Camassa & Holm [11], Constantin et al [12-15], Lenells [16- 18], Wazwaz [19, 20], Liu and Wang [21]. When $\gamma = 0$, Eq.(1.1) becomes the BBM equation [22], a well-known model for surface waves in a channel.

Recently, bifurcation method of dynamical systems has been used to investigate the nonlinear waves of some partial differential equations; see for instance Li and Liu [23, 24], Liu et al [25- 27], Tang et al [28, 29].

In this paper, we employ the bifurcation method of dynamical systems and numerical exploration to study the periodic waves of Eq.(1.1) with parameter $\gamma < 0$. Firstly, we derive travelling wave equation and system. Then we draw bifurcation curves and bifurcation phase portraits of the travelling wave system. From the bifurcation phase portraits one can see all the closed orbits. By using these closed orbits, the implicit expressions or explicit expressions of periodic waves are obtained. Our work extends previous results.

Specially we find another new phenomenon in Eq.(1.1): When the initial value tends to a certain number, the periodic shock wave suddenly changes into a smooth periodic wave. In dynamical systems, this represents that one of orbits can pass through the singular line (see orbit $\overline{\Gamma}$ in Fig.1 (I)). We give not only theoretical derivation, but also numerical simulation. The coherency of the consequences deduced from these two methods implies the correctness of our conclusions.

This paper is organized as follows. In Section 2, we give a preliminary. Our main results and two sets of graphs of implicit functions and explicit functions are given in Section 3. We arrange the theoretic derivation of our main results in Section 4. In Section 5, two sets of numerical simulations are displayed to test the correctness of our theoretic derivation. A short conclusion is also given in this section.

2. Preliminary

In this section, firstly we derive travelling wave equation and system. Then we draw the bifurcation curves and bifurcation phase portraits of the travelling wave system. These phase portraits will be used to obtain our main results.