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GLOBAL REGULARITY IN HOMOGENEOUS MORREY-HERZ SPACES OF SOLUTIONS TO NONDIVERGENCE ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS*

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Abstract In this paper, by establishing the boundedness results of singular integral operators and linear commutators, we obtain the global regularity, in homogeneous Morrey-Herz spaces, of strong solutions to nondivergence elliptic equations with VMO coefficients.

Key Words Elliptic equation; Morrey-Herz space; VMO; singular integral; commutator.

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1. Introduction

Let Ω be an open subset of $\mathbf{R}^n$. Let $B_k = B(0, 2^k) = \{x \in \mathbf{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbf{Z}$. Let $\mathcal{X}_k = \mathcal{X}_{A_k}$ be the characteristic function of the set A_k for $k \in \mathbf{Z}$.

Definition 1.1 *Let $\alpha \in \mathbf{R}, 0 < p \leq \infty, 0 < q < \infty$, and $\lambda \geq 0$. The homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$ are defined by*

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega) = \{f \in L^q_{loc}(\Omega \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} = \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\mathcal{X}_k\|^p_{L^q(\Omega)} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

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The more general spaces are first introduced recently by Lu and Xu in [1]. Compare the homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ with the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\Omega)$ (see [2]) and the Morrey spaces $M_q^{\lambda}(\Omega)$ (see [3]), where $\dot{K}_q^{\alpha,p}(\Omega)$ is defined by

$$\dot{K}_q^{\alpha,p}(\Omega) = \{f \in L_{loc}^q(\Omega \setminus 0) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\mathcal{X}_k\|_{L^q(\Omega)}^p < \infty\},$$

and $M_q^{\lambda}(\Omega)$ is defined by

$$M_q^{\lambda}(\Omega) = \{f \in L_{loc}^q(\Omega) : \sup_{\rho>0, x\in\Omega} \frac{1}{\rho^{\lambda}} \int_{B(x,\rho)\cap\Omega} |f(y)|^q \, dy < \infty\}.$$

It is easy to see that $M\dot{K}_{p,q}^{\alpha,0}(\Omega) = \dot{K}_q^{\alpha,p}(\Omega)$ and $M_q^{\lambda}(\Omega) \subset M\dot{K}_{q,q}^{0,\lambda}(\Omega)$.

Assume $f \in M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, the main purpose of this paper is to investigate the regularity, in the homogeneous Morrey-Herz spaces, of the strong solutions to the following Dirichlet problem on the second-order elliptic equations in nondivergence form:

$$\begin{cases} Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} = f \quad \text{a.e. in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, Ω is a bounded domain $C^{1,1}$ of $\mathbf{R}^n$, the coefficients $\{a_{ij}\}_{i,j=1}^n$ are symmetric and uniformly elliptic in Ω , i.e., for some $\Lambda > 0$ and every $\xi \in \mathbf{R}^n$,

$$a_{ij}(x) = a_{ji}(x), \quad \text{and} \quad \Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1.2}$$

with a.e. $x \in \Omega$. Moreover, we assume that $a_{ij}(x) \in VMO(\Omega)$, the spaces of the functions of vanishing mean oscillation introduced by Sarason in [4].

By establishing the boundedness results of singular integral operators and linear commutators (see Section 2), owing to the integral representation formulas given in [5, 6], we obtain the following global regularity, in homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, of the strong solution u to (1.1):

$$\|u\|_{W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} \le C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)}, \tag{1.3}$$

where $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, $-n/q+\lambda \le \alpha \le n(1-1/q)+\lambda$, and $W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ is the homogeneous Sobolev-Merry-Herz spaces defined in Section 3. We can see that, when $\alpha = 0$ and $p = q$, the above results coordinate with those in the setting of the Merry spaces $M_q^{\lambda}(\Omega)$, which are proved by Fan, Lu, and Yang in [7].