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EXISTENCE AND MULTIPLICITY RESULTS FOR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT AND HARDY TERM*

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Dedicated to Professor Guo Boling on the occasion of his 70th birthday

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Abstract This paper concerns the existence and multiplicity of solutions for some semilinear elliptic equations with critical Sobolev exponent, Hardy term and the sublinear nonlinearity at origin. By using Ekeland's variational principle, we conclude the existence of nontrivial solution for this problem, the Clark's critical point theorem is used to prove the existence of infinitely many solutions for this problem with odd nonlinearity.

Key Words Critical Sobolev exponent; Brezis-Lieb lemma; genus; Hardy term; infinitely many solutions; nontrivial solution.

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1. Introduction and Main Results

In this paper, we consider the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*-2}u + \lambda f(x,u), & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $R^N (N \geq 3)$ with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $0 \leq \mu < \overline{\mu} \overset{\triangle}{=} (\frac{N-2}{2})^2$, $\lambda > 0$ and $f \in C(\overline{\Omega} \times R, R)$ satisfies the following conditions:

(f_1) $\lim\limits_{|t|\to\infty} \dfrac{f(x,t)}{|t|^{2^*-1}} = 0$ uniformly for $x \in \Omega$;

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(f_2) $\lim\limits_{|t|\to 0}\dfrac{f(x,t)}{t}=+\infty$ uniformly for $x\in\Omega$;

(f_3) $f(x,-t)=-f(x,t)$ for all $t\in R$ and $x\in\Omega$.

By Hardy inequality (see [1]):

$$\int_\Omega \frac{|u|^2}{|x|^2}\mathrm{d}x \le \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^2\mathrm{d}x, \quad \forall u\in H_0^1(\Omega),$$

we easily derive that for $0\le\mu<\overline{\mu}$, $\|u\|=\left(\int_\Omega\left(|\nabla u|^2-\mu\frac{|u|^2}{|x|^2}\right)\mathrm{d}x\right)^{\frac{1}{2}}$ is a norm in $H_0^1(\Omega)$, which is equivalent to the usual norm $(\int_\Omega|\nabla u|^2\mathrm{d}x)^{\frac{1}{2}}$, and the best Hardy-Sobolev constant is defined by

$$S_\mu=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega\left(|\nabla u|^2-\mu\frac{|u|^2}{|x|^2}\right)\mathrm{d}x}{\left(\displaystyle\int_\Omega|u|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}>0.$$

The energy function corresponding to (1) is given by

$$I(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)\mathrm{d}x-\frac{1}{2^*}\int_\Omega|u|^{2^*}\mathrm{d}x-\lambda\int_\Omega F(x,u)\mathrm{d}x,$$

where $F(x,t)$ is a primitive function of $f(x,t)$ defined by $F(x,t)=\int_0^t f(x,s)\mathrm{d}s$ for $x\in\Omega$, $t\in R$. It follows from Hardy inequality and $f\in C(\overline{\Omega}\times R,R)$ that $I\in C^1(H_0^1(\Omega),R)$. Now it is well known that there exists one to one correspondence between the weak solutions of (1) and the critical points of I on $H_0^1(\Omega)$. More precisely we say that $u\in H_0^1(\Omega)$ is a weak solution of (1), if for any $v\in H_0^1(\Omega)$, there holds

$$\langle I'(u),v\rangle=\int_\Omega\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)\mathrm{d}x-\int_\Omega|u|^{2^*-2}uv\mathrm{d}x-\lambda\int_\Omega f(x,u)v\mathrm{d}x=0. \qquad (2)$$

In recent years, much attention has been paid to the special case of (1), with $f(x,u)\equiv u$:

$$\begin{cases}-\Delta u-\mu\frac{u}{|x|^2}=|u|^{2^*-2}u+\lambda u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \qquad (3)$$

Jannelli [2] considers the problem (3) and prove that the problem (3) admits a positive solution for $0<\mu\le\overline{\mu}-1$ and $\lambda\in(0,\lambda_1(\mu))$; for $\overline{\mu}-1<\mu<\overline{\mu}$ and $\Omega=B_1(0)$, there exists $\lambda^*\in(0,\lambda_1(\mu))$ such that the problem (3) admits a positive solution if and only if $\lambda\in(\lambda^*,\lambda_1(\mu))$, where $\lambda_1(\mu)$ is the first eigenvalue of $-\Delta-\frac{\mu}{|x|^2}$ in $H_0^1(\Omega)$. Cao and Peng [3] also consider the problem (3) and prove that for $N\ge 7$, $\mu\in[0,\overline{\mu}-4)$, the problem (3) possesses at least a pair of sign-changing solutions for any $\lambda\in(0,\lambda_1(\mu))$. Cao and Han [4] prove that for $\mu\in\left[0,\overline{\mu}-(\frac{N+2}{N})^2\right)$ the problem (3) admits a nontrivial