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GLOBAL WEAKLY DISCONTINUOUS SOLUTIONS TO A KIND OF MIXED INITIAL-BOUNDARY VALUE PROBLEM FOR INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEMS*

Guo Fei
(School of Mathematical Sciences, Fudan University, Shanghai 200433; Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210046, China)
(E-mail: gmath@126.com)
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Abstract In this paper we study the mixed initial-boundary value problem for inhomogeneous quasilinear hyperbolic systems in the domain $D = \{(t, x) \mid t \geq 0, x \geq 0\}$. Under the assumption that the source term satisfies the matching condition, a sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solution is given.

Key Words Inhomogeneous quasilinear hyperbolic system; mixed initial-boundary value problem; global weakly discontinuous solution; weak linear degeneracy; matching condition.

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1. Introduction and Main Results

Let us consider the following first order inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u)$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, and $B(u)$ is a vector function with suitably smooth elements $b_i(u)$ $(i = 1, \cdots, n)$ and satisfies

$$B(0) = 0,\ \nabla B(0) = 0. \tag{1.2}$$

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)$ $(i,j=1,\cdots,n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i,j=1,\cdots,n)$.

Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i,j=1,\cdots,n), \tag{1.5}$$

where δ_{ij} denotes the Kronecker's symbol.

For the Cauchy problem of homogeneous system (1.1) (i.e., $B(u) \equiv 0$) with the initial data

$$t=0: \quad u=\phi(x), \tag{1.6}$$

where $\phi(x)$ is a suitably smooth vector function, many results have been established (cf. [1–5] and references therein). Li and Wang in their recent work [6] obtained a necessary and sufficient condition to guarantee the existence and uniqueness of global weakly discontinuous solution to the Cauchy problem (1.1) with a kind of non-smooth initial data. If $B(u)$ satisfies the matching condition, we have generalized their results to the inhomogeneous case (cf. [7]).

Suppose

$$\lambda_1(0),\cdots,\lambda_m(0) < 0 < \lambda_{m+1}(0) < \cdots < \lambda_n(0). \tag{1.7}$$

On the domain $D=\{(t,x) \mid t\geq 0, x\geq 0\}$, we consider the mixed initial-boundary value problem (MIBVP, in short) (1.1) with initial data (1.6) and the nonlinear boundary conditions

$$x=0: \quad v_s = f_s(\alpha(t),v_1,\cdots,v_m)+h_s(t) \ (s=m+1,\cdots,n), \tag{1.8}$$

where

$$v_i = l_i(u)u \ (i=1,\cdots,n) \tag{1.9}$$

and

$$\alpha(t) = (\alpha_1(t),\cdots,\alpha_p(t)). \tag{1.10}$$

Without loss of generality, we suppose

$$f_s(\alpha(t),0,\cdots,0) \equiv 0 \quad (s=m+1,\cdots,n). \tag{1.11}$$

In the homogeneous case $B(u) \equiv 0$ it is proved that, if $\lambda_{m+1}(u),\cdots,\lambda_n(u)$ are weakly linearly degenerate and the conditions of C^1 compatibility are satisfied at the origin $(0,0)$, then MIBVP (1.1), (1.6) and (1.8) admits a unique global classical C^1 solution on D for some small and decaying data (cf. [8]). On the other hand, if only the conditions of C^0 compatibility are required at the origin $(0,0)$, under the same assumptions as in [8], we proved in [9] that MIBVP (1.1), (1.6) and (1.8) admits a unique global weakly discontinuous solution on the domain D.

Our goal in this article is to generalize the result in [9] to the inhomogeneous case. Under the assumption that $B(u)$ satisfies the matching condition (see Def. 1.1), we will